PART B

Unit - VI

Numerical Methods - 2

6.1 Introduction

We come across with situations, while solving an engineering / physical problem or in the course of an experiment, having a data comprising a set of discrete values of the dependent variable corresponding to various values (equidistant or otherwise) of the independent variable.

We discuss various ***numerical methods*** for

(i) Estimating to a desired degree of accuracy the value of the dependent variable corresponding to a value of the independent variable

(ii) Knowing the form of function which satisfy all the conditions / data on hand

(iii) Obtaining the derivatives of the unknown function at some specified points of the independent variable

(iv) Obtaining the definite integral of the unknown function or the value of the definite integral without the actual integration.

The numerical methods are highly useful when the analytical / theoretical approach to the problems are either unavailable or highly difficult.

6.2 Finite differences

Consider a function $y = f(x)$. Let x_0 , $x_1 = x_0 + h$, $x_2 = x_1 + h$,. . ., $x_n = x_{n-1} + h$ be a set of points at a common interval h. Let the corresponding values of $y = f(x)$ be $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$,. . ., $y_n = f(x_n)$.

The value of the independent variable x is called the *argument* and the corresponding functional value is known as *entry*. We define **forward** and **backward differences** concerning these values.

6.21 Forward Differences

The *first forward difference* of $f(x)$ denoted by $\Delta f(x)$ is defined as follows.

$$\Delta f(x) = f(x+h) - f(x)$$

Δ is called the *forward difference operator*.

Thus we have for the values x_0 , $x_1, x_2, \dots, x_n$:

$$\Delta f(x_0) = f(x_0+h)-f(x_0) \quad \text{or} \quad \Delta y_0 = y_1 - y_0$$

$$\Delta f(x_1) = f(x_1+h)-f(x_1) \quad \text{or} \quad \Delta y_1 = y_2 - y_1$$

$$\Delta f(x_2) = f(x_2+h)-f(x_2) \quad \text{or} \quad \Delta y_2 = y_3 - y_2 \text{ etc.,}$$

$$\Delta f(x_{n-1}) = f(x_{n-1}+h)-f(x_{n-1}) \quad \text{or} \quad \Delta y_{n-1} = y_n - y_{n-1}$$

The difference of the first forward differences are called *second forward differences*. They are as follows.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 , \quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1 ,$$

$$\Delta^2 y_2 = \Delta y_3 - \Delta y_2 , \dots \Delta^2 y_{n-2} = \Delta y_{n-1} - \Delta y_{n-2}$$

Similarly the other higher order differences namely the third, fourth, etc., are obtained and tabulated. Such a tabular arrangement is called **forward difference table** :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\dots$	$\Delta^n y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		.	
.	.	.	.	.	.	$\Delta^n y_0$
.	.	.	.		.	
.	.	.	.	$\Delta^3 y_{n-3}$		
x_{n-1}	y_{n-1}		$\Delta^2 y_{n-2}$			
		Δy_{n-1}				
x_n	y_n					

The first entries in the table namely Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, $\cdots$ $\Delta^n y_0$ are called the *leading forward differences.*

6.22 Backward differences

The *first backward difference* of $f(x)$ denoted by $\nabla f(x)$ is defined as follows.

$$\nabla f(x) = f(x) - f(x-h)$$

∇ is called the *backward difference operator.*

If $\quad x = x_n : \nabla f(x_n) = f(x_n) - f(x_n - h) \quad$ or $\quad \nabla y_n = y_n - y_{n-1}$

$\because \quad x_n - h = x_{n-1}$ and $f(x_{n-1}) = y_{n-1}$

If $\quad x = x_{n-1} : \nabla f(x_{n-1}) = f(x_{n-1}) - f(x_{n-1} - h)$

or $\quad \nabla y_{n-1} = y_{n-1} - y_{n-2}$ etc. $\nabla y_2 = y_2 - y_1$, $\nabla y_1 = y_1 - y_0$

The difference of the first backward differences are known as *second backward differences.* They are as follows.

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}, \quad \nabla^2 y_{n-1} = \nabla y_{n-1} - \nabla y_{y-2}, \ldots,$$

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

Similarly the other higher order backward differences namely the third, fourth etc., are formed and tabulated. Such a tabular arrangement is called **backward difference table** :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\ldots$	$\nabla^n y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$	.	.	
.	.	.	.	.	.	$\nabla^n y_n$
.	.	.	.	.	.	
.	.	.	.	$\nabla^3 y_n$		
x_{n-1}	y_{n-1}		$\nabla^2 y_n$			
		∇y_n				
x_n	y_n					

The last entries in the table namely $\nabla y_n, \nabla^2 y_n, \nabla^3 y_n, \cdots \nabla^n y_n$ are called the *leading backward differences.*

WORKED PROBLEMS

1. *Construct a finite difference table for the function* $f(x) = x^3 + x + 1$ *where* x *takes the values* 0, 1, 2, 3, 4, 5, 6. *Identify the leading forward and backward differences.*

>> $f(x) = x^3 + x + 1$ by data. From this we obtain

$f(0) = 1, f(1) = 3, f(2) = 11, f(3) = 31, f(4) = 69, f(5) = 131, f(6) = 223.$

The finite difference table is as follows.

x	$f(x) = y$	First difference	Second difference	Third difference	Fourth difference
0	1				
		2			
1	3		6		
		8		6	
2	11		12		0
		20		6	
3	31		18		0
		38		6	
4	69		24		0
		62		6	
5	131		30		
		92			
6	223				

Taking $x_0 = 0$, $y_0 = 1$, the first value in every column are the leading forward differences. They are as follows :

$$\Delta y_0 = 2, \quad \Delta^2 y_0 = 6, \quad \Delta^3 y_0 = 6, \quad \Delta^4 y_0 = 0$$

Also by taking $x_n = 6$, $y_n = 223$, the last value in every column are the leading backward differences. They are as follows :

$$\nabla y_n = 92, \quad \nabla^2 y_n = 30, \quad \nabla^3 y_n = 6, \quad \nabla^4 y_n = 0$$

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2. *Construct the table of differences for the following data*

x	0	10	20	30	40
$f(x)$	1	1.5	2.2	3.1	4.6

Evaluate $\Delta^2 f(20)$, $\Delta^3 f(10)$ *and* $\Delta^4 f(0)$

>> We shall construct the forward difference table.

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 0$	$y_0 = 1$				
		$\Delta y_0 = 0.5$			
$x_1 = 10$	$y_1 = 1.5$		$\Delta^2 y_0 = 0.2$		
		$\Delta y_1 = 0.7$		$\Delta^3 y_0 = 0$	
$x_2 = 20$	$y_2 = 2.2$		$\Delta^2 y_1 = 0.2$		$\Delta^4 y_0 = 0.4$
		$\Delta y_2 = 0.9$		$\Delta^3 y_1 = 0.4$	
$x_3 = 30$	$y_3 = 3.1$		$\Delta^2 y_2 = 0.6$		
		$\Delta y_3 = 1.5$			
$x_4 = 40$	$y_4 = 4.6$				

Thus we have from the table,

$$\Delta^2 f(20) = \Delta^2 f(x_2) = \Delta^2 y_2 = 0.6$$
$$\Delta^3 f(10) = \Delta^3 f(x_1) = \Delta^3 y_1 = 0.4$$
$$\Delta^4 f(0) = \Delta^4 f(x_0) = \Delta^4 y_0 = 0.4$$

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3. *Starting from the definition of the finite differences obtain expressions for* $\Delta^2 y_0$, $\Delta^3 y_0$, $\nabla^2 y_n$ *and* $\nabla^3 y_n$ *interms of the values of* y.

>> We have by the definition $\Delta f(x) = f(x+h) - f(x)$

$\therefore \quad \Delta f(x_0) = f(x_0 + h) - f(x_0)$ or $\Delta y_0 = y_1 - y_0$

Similarly $\Delta y_1 = y_2 - y_1$

Now $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$

Thus $\Delta^2 y_0 = y_2 - 2y_1 + y_0$

Now $\Delta^3 y_0 = \Delta(\Delta^2 y_0) = \Delta(y_2 - 2y_1 + y_0)$

$= (y_3 - y_2) - 2(y_2 - y_1) + (y_1 - y_0)$

Thus $\mathbf{\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0}$

We have by the definition $\nabla y_n = y_n - y_{n-1}$

$\therefore \quad \nabla^2 (y_n) = \nabla y_n - \nabla y_{n-1}$

$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$

Thus $\mathbf{\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}}$

Next $\nabla^3 y_n = \nabla(\nabla^2 y_n) = \nabla(y_n - 2y_{n-1} + y_{n-2})$

$= (y_n - y_{n-1}) - 2(y_{n-1} - y_{n-2}) + (y_{n-2} - y_{n-3})$

Thus $\mathbf{\nabla^3 y_n = y_n - 3y_{n-1} + 3y_{n-2} + y_{n-3}}$

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4. *Prove the following results.*

(*a*) $y_3 = y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0$

(*b*) $\nabla^2 y_8 = y_8 - 2y_7 + y_6$

>> (a) Consider R.H.S = $y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0$

i.e., $= y_2 + (y_2 - y_1) + (y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0) = y_3$

Thus **R.H.S = L.H.S**

(b) $\nabla y_8 = y_8 - y_7$

$\therefore \quad \nabla(\nabla y_8) = \nabla y_8 - \nabla y_7 = (y_8 - y_7) - (y_7 - y_6)$

Thus $\mathbf{\nabla^2 y_8 = y_8 - 2y_7 + y_6}$

Obtain the following :

5. $\Delta\,(x + \cos x)$

6. $\Delta^n\,(e^{2x+3})$

7. $\Delta^n\,(a\,b^x)$

8. $\Delta^2 \left[\dfrac{1}{x^2 + 5x + 6} \right]$ *where* $h = 1$

9. $\Delta \left[\tan^{-1} \left(\dfrac{n-1}{n} \right) \right]$ *where* $h = 1$

>> 5. $\Delta(x+\cos x) = \{(x+h)+\cos(x+h)\} - (x+\cos x)$

$$= h + \{\cos(x+h) - \cos x\}$$

$$= h - 2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$\because$ $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Thus $\boldsymbol{\Delta(x+\cos x) = h - 2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}$

>> **6.** $\Delta(e^{2x+3}) = e^3\,\Delta(e^{2x}) = e^3\left\{e^{2(x+h)} - e^{2x}\right\}$

$$= e^3 e^{2x}(e^{2h}-1)$$

$\therefore$ $\Delta(e^{2x+3}) = e^{2x+3}(e^{2h}-1)$

Now $\Delta(\Delta e^{2x+3}) = (e^{2h}-1)\,\Delta(e^{2x+3})$

i.e., $\Delta^2(e^{2x+3}) = (e^{2h}-1)\left\{e^{2x+3}(e^{2h}-1)\right\}$

or $\Delta^2(e^{2x+3}) = (e^{2h}-1)^2 e^{2x+3}$ and so on.

Thus we get, $\boldsymbol{\Delta^n(e^{2x+3}) = (e^{2h}-1)^n\, e^{2x+3}}$

>> **7.** $\Delta(a\,b^x) = a\,b^{x+h} - a\,b^x = a\,b^x(b^h-1)$

Also $\Delta\,\Delta(a\,b^x) = (b^h-1)\,\Delta(a\,b^x)$

i.e., $\Delta^2(a\,b^x) = (b^h-1)\{a\,b^x(b^h-1)\}$

or $\Delta^2(a\,b^x) = (b^h-1)^2 a\,b^x$ and so on.

Thus we get, $\boldsymbol{\Delta^n(a\,b^x) = (b^h-1)^n\, a\,b^x}$

>> **8.** Let $f(x) = \dfrac{1}{x^2+5x+6} = \dfrac{1}{(x+2)(x+3)}$

$\Delta f(x) = f(x+h) - f(x)$ by the definition.

Since $h = 1,\ \Delta f(x) = f(x+1) - f(x)$

Now, $\Delta\{\Delta f(x)\} = \Delta f(x+1) - \Delta f(x)$

$$= \{f(x+2) - f(x+1)\} - \{f(x+1) - f(x)\}$$

$\therefore$ $\Delta^2 f(x) = f(x+2) - 2f(x+1) + f(x)$

$$\Delta^2 f(x) = \frac{1}{(x+4)(x+5)} - \frac{2}{(x+3)(x+4)} + \frac{1}{(x+2)(x+3)}$$

$$= \frac{(x+2)(x+3) - 2(x+2)(x+5) + (x+4)(x+5)}{(x+2)(x+3)(x+4)(x+5)}$$

$$= \frac{(x^2+5x+6) - (2x^2+14x+20) + (x^2+9x+20)}{(x+2)(x+3)(x+4)(x+5)}$$

Thus $\Delta^2 \left[\dfrac{1}{x^2+5x+6} \right] = \dfrac{6}{(x+2)(x+3)(x+4)(x+5)}$ where $h = 1$

>> **9.** Let $f(n) = \tan^{-1}\left(\dfrac{n-1}{n}\right) = \tan^{-1}\left(1 - \dfrac{1}{n}\right)$

$\Delta f(n) = f(n+1) - f(n)$ since $h = 1$ by data.

$$= \tan^{-1}\left(1 - \frac{1}{n+1}\right) - \tan^{-1}\left(1 - \frac{1}{n}\right)$$

But $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

$$\therefore \quad \Delta f(n) = \tan^{-1}\left[\frac{\frac{1}{n} - \frac{1}{n+1}}{1 + \left(1 - \frac{1}{n+1}\right)\left(1 - \frac{1}{n}\right)}\right]$$

$$\Delta f(n) = \tan^{-1}\left[\frac{\frac{1}{n(n+1)}}{2 - \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n(n+1)}}\right]$$

$$= \tan^{-1}\left[\frac{1}{2n(n+1) - (n+1) - n + 1}\right] = \tan^{-1}\left(\frac{1}{2n^2}\right)$$

Thus $\Delta \tan^{-1}\left(\dfrac{n-1}{n}\right) = \tan^{-1}\left(\dfrac{1}{2n^2}\right)$ where $h = 1$

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10. *Show that* $\Delta \log f(x) = \log\left\{1 + \dfrac{\Delta f(x)}{f(x)}\right\}$

>> L.H.S $= \Delta \log f(x) = \log f(x+h) - \log f(x)$

$$\text{L.H.S} = \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{f(x)+f(x+h)-f(x)}{f(x)}\right]$$

$$\text{L.H.S} = \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right] = \log\left[1+\frac{\Delta f(x)}{f(x)}\right] = \text{R.H.S}$$

Thus **L.H.S = R.H.S**

6.3 Interpolation

If $y_0, y_1, y_2, \cdots y_n$ be a set of values of an unknown function $y = f(x)$ corresponding to the values of $x: x_0, x_1, x_2, \cdots x_n$, the process of finding (*estimating*) the value of y for any given value of x between x_0 and x_n is called ***interpolation***. Also the process of finding (*estimating*) the value of y outside the given range of x is called ***extrapolation***. In general the concept of interpolation includes extrapolation also.

Thus we can say that interpolation is a technique of estimating the value of an unknown function for any intermediate value of the independent variable.

For example, if we have the data :

x	0	4	5	8	10	15
$f(x)$	6	15	17	29	40	87

the process of estimating $f(3)$, $f(7)$, $f(12.5)$, $f(14)$ etc.is interpolation and estimating $f(-0.5)$, $f(18)$, $f(25)$ etc. is extrapolation.

We first discuss interpolation for equal intervals which will be followed with interpolation for unequal intervals.

6.4 Interpolation formulae for equal intervals/equidistant arguments

We discuss interpolation formulae for equal intervals based on forward and backward differences.

These formulae are established by approximating the unknown function to a polynomial in x whose values coincide with the value of $f(x)$ at the specified points of $x: x_0, x_1, x_2, \cdots x_n$.

Forward difference interpolation formula and Backward difference interpolation formula

Let $y_0, y_1, y_2, \cdots y_n$ be the values of an unknown function $y = f(x)$ corresponding to equidistant values of $x: x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \cdots x_n = x_0 + nh$.

Then we have the following two interpolation formulae.

6.41 Newton's forward interpolation formula

The value of $y = f(x)$ at $x = x_0 + rh$, that is $y_r = f(x_0 + rh)$ is approximately given by

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots + \frac{r(r-1)(r-2)\cdots(r-\overline{n-1})}{n!}\Delta^n y_0$$

where r is any real number.

6.42 Newton's backward interpolation formula

The value of $y = f(x)$ at $x = x_n + rh$, that is $y_r = f(x_n + rh)$ is approximately given by

$$y_r = y_n + r\,\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots + \frac{r(r+1)(r+2)\cdots(r+\overline{n-1})}{n!}\nabla^n y_n$$

where r is any real number.

Appropriate interpolation formula

To estimate the value of y at a desired value of x near the beginning of the table (*first half, x is close to x_0*) forward formula is appropriate.

Similarly to estimate the value of y at a desired value of x near the end of the table (*second half, x is close to x_n*) backward formula is appropriate.

The polynomial $y = f(x)$ satisfying the data can also be found from these formulae and it is called an interpolating polynomial.

Note : *The word appropriate is used in the sense that the computational work will involve relatively smaller magnitudes. Either of the formulae can be used for obtaining the required result.*

Working procedure for problems

- We construct the difference table in accordance with the interpolation formula.
- We compute the value of r where

(a) $r = \dfrac{x - x_0}{h}$ in the case of forward interpolation formula, x_0 being the first value of x and h is the step length.

(b) $r = \dfrac{x - x_n}{h}$ in the case of backward interpolation formula, x_n being the last value of x and h is the step length.

➲ The value of r along with the value of the finite differences are substituted in the interpolation formula which results in the value of y at the desired value of x.

WORKED PROBLEMS

11. *Find* $y(1.4)$ *given the data*

x	1	2	3	4	5
y	10	26	58	112	194

>> Here we have to find y at $x = 1.4$

Since the value $x = 1.4$ is in the first half of the table near $x = 1$, Newton's forward interpolation formula is appropriate and we shall construct the forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 1$	$y_0 = 10$				
		16			
2	26		16		
		32		6	
3	58		22		0
		54		6	
4	112		28		
		82			
5	194				

We have Newton's forward interpolation formula :

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\,\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\,\Delta^3 y_0 + \cdots$$

where $r = \dfrac{x - x_0}{h}$

x = point at which y is required = 1.4

x_0 = initial point (*first value of* x) = 1

h = common interval length = 1. Here $r = \dfrac{1.4 - 1}{1} = 0.4$

From the table $\Delta y_0 = 16$, $\Delta^2 y_0 = 16$, $\Delta^3 y_0 = 6$, $\Delta^4 y_0 = 0$

$$\therefore \quad y(1.4) = f(1.4) = 10 + (0.4)16 + \frac{(0.4)(0.4-1)}{2}(16) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(6)$$

$$= 10 + 6.4 + (0.4)(-0.6)8 + (0.4)(-0.6)(-1.6) = 14.864$$

Thus $\boldsymbol{y(1.4) = 14.864}$

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12. *Find $u_{0.5}$ from the data :*

$$u_0 = 225, u_1 = 238, u_2 = 320, u_3 = 340$$

>> The value $x = 0.5$ is near to $x = 0$ and hence Newton's forward interpolation formula is appropriate.

We shall first construct the forward difference table.

x	$u_x = y$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$y_0 = 225$			
		$\Delta y_0 = 13$		
1	238		$\Delta^2 y_0 = 69$	
		82		$\Delta^3 y_0 = -131$
2	320		-62	
		20		
3	340			

We have Newton's forward interpolation formula :

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots$$

where $r = \dfrac{x - x_0}{h}$; $r = \dfrac{0.5 - 0}{1} = 0.5$

$$\therefore \quad f(0.5) = 225 + 0.5(13) + \frac{(0.5)(0.5-1)}{2}(69) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(-131) = 214.6875$$

Thus $\boldsymbol{u_{0.5} = 214.6875}$

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13. *The area of a circle (A) corresponding to diameter (D) is given below.*

D	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find the area corresponding to diameter 105 using an appropriate interpolation formula.

>> Here we have to find A when $D = 105$

As this value 105 is near to the end value 100, Newton's backward interpolation formula is appropriate. D and A correspond to x and y. The backward difference table is formed first.

$x = D$	$y = A$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		−2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
$x_n = 100$	$y_n = 7854$				

We have Newton's backward interpolation formula :

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

where $r = \dfrac{x - x_n}{h}$; $r = \dfrac{105 - 100}{5} = 1$

From the table $\nabla y_n = 766$, $\nabla^2 y_n = 40$, $\nabla^3 y_n = 2$, $\nabla^4 y_n = 4$

$$\therefore \quad f(105) = 7854 + 1(766) + \frac{(1)(2)}{2}(40) + \frac{(1)(2)(3)}{6}(2) + \frac{(1)(2)(3)(4)}{24}(4)$$

$$= 7854 + 766 + 40 + 2 + 4 = 8666$$

Thus **the area (A) corresponding to diameter (D) = 105 is 8666**

14. *The following table give the values of* tan x *for* $0.10 \le x \le 0.30$. *Find tan* (0.26)

x	0.10	0.15	0.20	0.25	0.30
tan x	0.1003	0.1511	0.2027	0.2553	0.3093

>> Here the value $x = 0.26$ is near the end value 0.30

Hence Newton's backward interpolation formula is appropriate. The backward difference table is as follows.

x	$f(x) = y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
$x_n = 0.30$	$y_n = 0.3093$				

We have Newton's backward interpolation formula :

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

where $r = \dfrac{x - x_n}{h}$; $r = \dfrac{0.26 - 0.30}{0.05} = -0.8.$

From the table,

$$\nabla y_n = 0.0540, \ \nabla^2 y_n = 0.0014, \ \nabla^3 y_n = 0.0004, \ \nabla^4 y_n = 0.0002$$

$$\therefore \quad f(0.26) = 0.3093 + (-0.8)(0.054) + \frac{(-0.8)(-0.8+1)}{2}(0.0014) + \frac{(-0.8)(-0.8+1)(-0.8+2)}{6}(0.0004) + \frac{(-0.8)(-0.8+1)(-0.8+2)(-0.8+3)}{24}(0.0002)$$

$$f(0.26) = 0.26602$$

Thus **tan (0.26) = 0.2660**

15. *Extrapolate for 25.4 given the data*

x	19	20	21	22	23
y	91	100.25	110	120.25	131

>> **We shall first construct the backward difference table.**

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
19	91			
		9.25		
20	100.25		0.5	
		9.75		0
21	110		0.5	
		10.25		$\nabla^3 y_n = 0$
22	120.25		$\nabla^2 y_n = 0.5$	
		$\nabla y_n = 10.75$		
$x_n = 23$	$y_n = 131$			

We have Newton's backward interpolation formula :

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

where $r = \dfrac{x - x_n}{h}$; $r = \dfrac{25.4 - 23}{1} = 2.4$

$$\therefore \qquad f(25.4) = 131 + (2.4)10.75 + \frac{(2.4)(2.4+1)}{2}(0.5) = 158.84$$

Thus $f(25.4) = \mathbf{158.84}$

---

16. *Given* $f(40) = 184, f(50) = 204, f(60) = 226, f(70) = 250, f(80) = 276, f(90) = 304$, *find* $f(38)$ *and* $f(85)$ *using suitable interpolation formulae.*

>> Here we shall find $f(38)$ using Newton's forward interpolation formula and find $f(85)$ using Newton's backward interpolation formula. The finite difference table applicable to both the interpolation formulae is as follows.

x	$f(x) = y$	I difference	II difference	III difference
40	184			
		20		
50	204		2	
		22		0
60	226		2	
		24		0
70	250		2	
		26		0
80	276		2	
		28		
90	304			

To find $f(38)$: We have from the table

$$x_0 = 40,\ y_0 = 184,\ \Delta y_0 = 20,\ \Delta^2 y_0 = 2,\ \Delta^3 y_0 = 0$$

We have Newton's forward interpolation formula :

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\,\Delta^2 y_0 + \cdots$$

where $r = \dfrac{x - x_0}{h}$; $r = \dfrac{38 - 40}{10} = -0.2$

$$f(38) = 184 + (-0.2)(20) + \frac{(-0.2)(-0.2-1)}{2}(2) = 180.24$$

Thus $f(38) = \mathbf{180.24}$

To find $f(85)$: We have from the table

$$x_n = 90,\ y_n = 304,\ \nabla y_n = 28,\ \nabla^2 y_n = 2,\ \nabla^3 y_n = 0$$

Also we have Newton's backward interpolation formula :

$$y_r = y_n + r\,\nabla y_n + \frac{r(r+1)}{2!}\,\nabla^2 y_n + \cdots$$

where $r = \dfrac{x - x_n}{h}$; $r = \dfrac{85 - 90}{10} = -0.5.$

$$f(85) = 304 + (-0.5)(28) + \frac{(-0.5)(-0.5+1)}{2}(2) = 289.75$$

Thus $f(85) = \mathbf{289.75}$

17. *Given* $\sin 45^0 = 0.7071$, $\sin 50^0 = 0.7660$, $\sin 55^0 = 0.8192$, $\sin 60^0 = 0.8660$, *find* $\sin 57^0$ *using an appropriate interpolation formula.*

>> We have to find the value of $f(x) = \sin x$ at $x = 57^0$ which is near the end value $x = 60^0$ and hence Newton's backward interpolation formula is appropriate. The
difference table is as follows.

x	$f(x) = y$	∇y	$\nabla^2 y$	$\nabla^3 y$
45	0.7071			
		0.0589		
50	0.7660		− 0.0057	
		0.0532		− 0.0007
55	0.8192		− 0.0064	
		0.0468		
$x_n = 60$	$y_n = 0.8660$			

We have Newton's backward interpolation formula :

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

where $r = \dfrac{x - x_n}{h}$; $r = \dfrac{57 - 60}{5} = -0.6.$

From the table, $\nabla y_n = 0.0468, \nabla^2 y_n = -0.0064, \nabla^3 y_n = -0.0007$

$$\therefore \quad f(57) = 0.8660 + (-0.6)(0.0468) + \frac{(-0.6)(-0.6+1)}{2}(-0.0064)$$
$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-0.0007) = 0.8387$$

Thus **sin 57° = 0.8387**

---

18. *Find the interpolating polynomial* $f(x)$ *satisfying* $f(0) = 0$, $f(2) = 4$, $f(4) = 56$, $f(6) = 204$, $f(8) = 496$, $f(10) = 980$ *and hence find* $f(3)$, $f(5)$ *and* $f(7)$

>> The interpolating polynomial can be found from either of the two interpolation formulae. We shall use Newton's forward interpolation formula. The difference table is as follows.

+0

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 0$	$y_0 = 0$				
		$\Delta y_0 = 4$			
2	4		$\Delta^2 y_0 = 48$		
		52		$\Delta^3 y_0 = 48$	
4	56		96		$\Delta^4 y_0 = 0$
		148		48	
6	204		144		0
		292		48	
8	496		192		
		484			
10	980				

We have Newton's forward interpolation formula :

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\,\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\,\Delta^3 y_0 + \cdots,$$

where $r = \dfrac{x - x_0}{h}$ Here $r = \dfrac{x-0}{2}$ or $r = \dfrac{x}{2}$

$$\therefore \quad y = f(x) = 0 + \frac{x}{2}(4) + \frac{\frac{x}{2}\left(\frac{x}{2} - 1\right)}{2}(48) + \frac{\frac{x}{2}\left(\frac{x}{2} - 1\right)\left(\frac{x}{2} - 2\right)}{6}(48)$$

$$= 2x + \frac{x}{2}\left(\frac{x-2}{2}\right)(24) + \frac{x}{2}\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)(8)$$

$$= 2x + x(x-2)(6) + x(x-2)(x-4)$$

$$= x\left[2 + 6(x-2) + (x^2 - 2x - 4x + 8)\right]$$

$$= x\left[2 + 6x - 12 + x^2 - 6x + 8\right] = x^3 - 2x$$

Thus **the interpolating polynomial is** $y = f(x) = x^3 - 2x$

Now putting $x = 3, 5, 7$ we obtain

$$f(3) = 21, \quad f(5) = 115, \quad f(7) = 329$$

Note : *It may be observed that if we substitute the given values of x namely 0, 2, 4, 6, 8, 10 in the polynomial, the values of* $f(x)$ *coincide with the values given in the data.*

- -

19. *From the following table find the number of students who have obtained (a) less than 45 marks (b) between 40 and 45 marks.*

Marks	30–40	40–50	50–60	60–70	70–80
No. of students	31	42	51	35	31

>> We shall reconstitute the given table with $f(x)$ representing the number of students less than x marks. That is,

less than 40 marks 31 students,

less than 50 marks $31 + 42 = 73$ students,

less than 60 marks $73 + 51 = 124$ students,

less than 70 marks $124 + 35 = 159$ students,

less than 80 marks $159 + 31 = 190$ students.

We have the new table along with the forward differences.

(a) We need to find $f(45)$ being the number of students scoring less than 45 marks.

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 40$	$y_0 = 31$				
		$\Delta y_0 = 42$			
50	73		$\Delta^2 y_0 = 9$		
		51		$\Delta^3 y_0 = -25$	
60	124		-16		$\Delta^4 y_0 = 37$
		35		12	
70	159		-4		
		31			
80	190				

(a) We shall find $f(45)$ by applying Newton's forward interpolation formula .

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

where $r = \dfrac{x - x_0}{h}$; $r = \dfrac{45 - 40}{10} = 0.5$

$$\therefore \quad f(45) = 31 + (0.5)42 + \frac{(0.5)(0.5-1)}{2}(9)$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)}{6}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(37)$$

$$f(45) = 47.86 \approx 48$$

Thus the number of students obtaining less than 45 marks is 48.

(b) We need to find $f(45) - f(40)$. But $f(40) = 31$ by data..

Hence $f9\,45) - f(40) = 48 - 31 = 17$

Thus the number of students scoring marks between 40 & 45 is 17.

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20. *A survey conducted in a slum locality reveals the following information as classified below.*

Income per day (Rs.)	Under 10	10-20	20-30	30-40	40-50
Number of persons	20	45	115	210	115

Estimate the probable number of persons in the income group 20 *to* 25.

>> The given data is reconstituted with $f(x)$ representing the number of persons less than income of Rs. x. That is,

less than Rs. 10 $= 20,$

less than Rs. 20 $= 20 + 45 = 65,$

less than Rs. 30 $= 65 + 115 = 180,$

less than Rs. 40 $= 180 + 210 = 390,$

less than Rs. 50 $= 390 + 115 = 505$

We have to find $f(25)$ and $f(20)$ which estimates the number of persons having income less than Rs.25 and less than Rs.20 so that their difference, that is $f(25) - f(20)$ will give us the number of persons having income between Rs.20 and 25. The new table along with the forward differences is as follows.

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 10$	$y_0 = 20$				
		45			
20	65		70		
		115		25	
30	180		95		− 215
		210		− 190	
40	390		− 95		
		115			
50	505				

We have Newton's forward interpolation formula :

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots$$

where $r = \dfrac{x - x_0}{h}$; $r = \dfrac{x - 10}{10}$

Case (i) : To find $f(25)$ we have $r = \dfrac{25-10}{10} = 1.5$

From the table $\Delta y_0 = 45,\ \Delta^2 y_0 = 70,\ \Delta^3 y_0 = 25,\ \Delta^4 y_0 = -215$

Now, $f(25) = 20 + 1.5(45) + \dfrac{(1.5)(1.5-1)}{2}(70) + \dfrac{(1.5)(1.5-1)(1.5-2)}{6}(25) + \dfrac{(1.5)(1.5-1)(1.5-2)(1.5-3)}{24}(-215)$

$\therefore\quad f(25) \approx 107$

Case (ii) : To find $f(20)$ we have $r = \dfrac{20-10}{10} = 1$

Now $f(20) = 20 + 1(45) + 0 + 0 + 0$

$\therefore\quad f(20) = 65$

Hence $f(25) - f(20) = 107 - 65 = 42$

Thus the number of persons in the income group of Rs.20 to 25 is 42.

21. *Compute* $u_{14.2}$ *from the following table by applying Newton's backward interpolation formula.*

x	10	12	14	16	18
u_x	0.240	0.281	0.318	0.352	0.384

>> We shall apply Newton's backward interpolation formula and the difference table is as follows.

x	$u_x = y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	0.240				
		0.041			
12	0.281		-0.004		
		0.037		0.001	
14	0.318		-0.003	$\nabla^3 y_n = 0.001$	$\nabla^4 y_n = 0$
16	0.352	0.034	$\nabla^2 y_n = -0.002$		
$x_n = 18$	$y_n = 0.384$	$\nabla y_n = 0.032$			

We have Newton's backward interpolation formula :

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

where $r = \dfrac{x - x_n}{h}$; $r = \dfrac{14.2 - 18}{2} = -1.9$

$$\therefore \quad u_{14.2} = 0.384 + (-1.9)(0.032) + \frac{(-1.9)(-0.9)}{2}(-0.002) + \frac{(-1.9)(-0.9)(0.1)}{6}(0.001)$$

Thus $\mathbf{u_{14.2} = 0.3215}$

---

22. *Use Newton's forward interpolation formula to find* y_{35} *given* $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$.

>> The difference table is as follows.

x	$y_x = y$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 20$	$y_0 = 512$			
		$\Delta y_0 = -73$		
30	439		$\Delta^2 y_0 = -20$	
		-93		$\Delta^3 y_0 = 10$
40	346		-10	
		-103		
50	243			

We have Newton's backward interpolation formula :

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

where $r = \dfrac{x - x_0}{h}$; $r = \dfrac{35-20}{10} = 1.5$

$$y_{35} = y(35) = 512 + (1.5)(-73) + \frac{(1.5)(0.5)}{2}(-20) + \frac{(1.5)(0.5)(-0.5)}{6}(10)$$

Thus $y_{35} = \mathbf{394.375}$

---

23. *Find $f(2.5)$ by using Newton's backward interpolation formula given that* $f(0) = 7.4720$ $f(1) = 7.5854$ $f(2) = 7.6922$ $f(3) = 7.8119$ $f(4) = 7.9252$

>> The backward difference table is as follows.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	7.4720				
		0.1134			
1	7.5854		-0.0066		
		0.1068		0.0195	
2	7.6922		0.0129		$\nabla^4 y_n = -0.0388$
		0.1197		$\nabla^3 y_n = -0.0193$	
3	7.8119		$\nabla^2 y_n = -0.0064$		
		$\nabla y_n = 0.1133$			
$x_n = 4$	$y_n = 7.9252$				

We have Newton's backward interpolation formula :

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

where $r = \frac{x - x_n}{h}$; $r = \frac{2.5-4}{1} = -1.5$

$$\therefore \quad f(2.5) = 7.9252 + (-1.5)(0.1133) + \frac{(-1.5)(-0.5)}{2}(-0.0064)$$

$$+ \frac{(-1.5)(-0.5)(0.5)}{6}(-0.0193) + \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(-0.0388)$$

Thus $f(2.5) = 7.7507$

---

24. *In a table given below, the values of y are consecutive terms of a series of which 23.6 is the 6^{th} term. Find the first and tenth terms of the series.*

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

>> We need to compute $y(1)$ and $y(10)$. Newton's forward and backward interpolation formula respectively will be appropriate.

We shall first construct the finite difference table.

x	y	I difference	II difference	III difference	IV difference
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

Case - (i) : To find $y(1)$

We have from the table,

$$x_0 = 3, \quad y_0 = 4.8, \quad \Delta y_0 = 3.6, \quad \Delta^2 y_0 = 2.5, \quad \Delta^3 y_0 = 0.5$$

We have Newton's forward interpolation formula :

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

where $r = \frac{x - x_0}{h}$; $r = \frac{1-3}{1} = -2$

$$y(1) = 4.8 + (-2)(3.6) + \frac{(-2)(-3)}{2}(2.5) + \frac{(-2)(-3)(-4)}{6}(0.5)$$

Thus $\mathbf{y(1) = 3.1}$

Case - (ii) : To find $y(10)$

We have from the table,

$$x_n = 9, \quad y_n = 73.9, \quad \nabla y_n = 21.1, \quad \nabla^2 y_n = 4.5, \quad \nabla^3 y_n = 0.5$$

We have Newton's backward interpolation formula :

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

where $r = \frac{x - x_n}{h}$; $r = \frac{10-9}{1} = 1$

$$y(10) = 73.9 + 1(21.1) + \frac{(1)(2)}{2!}(4.5) + \frac{(1)(2)(3)}{3!}(0.5)$$

Thus $\mathbf{y(10) = 100}$

---

25. *The population of a town is given by the table.*

year	1951	1961	1971	1981	1991
Population in thousands	19.96	39.65	58.81	77.21	94.61

Using Newton's forward and backward interpolation formula, calculate the increase in the population from the year 1955 to 1985

>> We shall first form the finite difference table.

x	y	I difference	II difference	III difference	IV difference
1951	19.96				
		19.96			
1961	39.65		− 0.53		
		19.16		− 0.23	
1971	58.81		− 0.76		− 0.01
		18.4		− 0.24	
1981	77.21		− 1		
		17.4			
1991	94.61				

Case - (i) : To find y (1955)

We have from the table,

$x_0 = 1951,\ y_0 = 19.96,\ \Delta y_0 = 19.69,\ \Delta^2 y_0 = -0.53\ \Delta^3 y_0 = -0.23,\ \Delta^4 y_0 = -0.01$

We have Newton's forward interpolation formula :

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\,\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\,\Delta^3 y_0$$
$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\,\Delta^4 y_0 + \cdots$$

where $r = \dfrac{x - x_0}{h}$; $r = \dfrac{1955 - 1951}{10} = 0.4$

$$y(1955) = 19.96 + (0.4)(19.69) + \frac{(0.4)(-0.6)}{2}(-0.53)$$
$$+ \frac{(0.4)(-0.6)(-1.6)}{6}(-0.23) + \frac{(0.4)(-0.6)(-1.6)(-2.6)}{24}(-0.01)$$

$\therefore$ $\quad$ $\mathbf{y(1955) = 27.89}$

Case - (ii) : To find y (1985)

We have from the table,

$x_n = 1991,\ y_n = 94.61,\ \nabla y_n = 17.4,\ \nabla^2 y_n = -1,\ \nabla^3 y_n = -0.24,\ \nabla^4 y_n = -0.01$

We have Newton's backward interpolation formula,

$$y_r = y_n + r\,\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

where $r = \dfrac{x - x_n}{h}$; $r = \dfrac{1985 - 1991}{10} = -0.6$

$$y(1985) = 94.61 + (-0.6)(17.4) + \frac{(-0.6)(0.4)}{2}(-1) + \frac{(-0.6)(0.4)(1.4)}{6}(-0.24) + \frac{(-0.6)(0.4)(1.4)(2.4)}{24}(-0.01)$$

$\therefore$ $\mathbf{y(1985) = 84.3}$

Thus the increase in population from the year 1955 to 1985 is 84.3 – 27.89 = 56.41 thousands.

EXERCISES

Find the following.

1. $\Delta(\tan^{-1} x)$
2. $\Delta^2(\cos 2x)$
3. $\Delta^n(e^x)$
4. $\Delta^2\left[\dfrac{5x+12}{x^2+5x+6}\right]$ where $h = 1$
5. $\Delta(\tan ax)$ where $h = 1$

Prove the following.

6. $y_4 = y_0 + 4\,\Delta y_0 + 6\,\Delta^2 y_0 + 4\,\Delta^3 y_0 + \Delta^4 y_0$
7. $\Delta^6 y_0 = y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0$
8. $\Delta^3 y_n = y_{n+3} - 3y_{n+2} + 3y_{n-1} - y_n$

Use an appropriate interpolation formula to estimate $y = f(x)$ for the given value of x [Problems 9 t0 12]

9.

x	1.7	1.8	1.9	2.0	2.1	2.2
$f(x)$	5.474	6.050	6.686	7.389	8.166	9.025

$f(1.85) = ?$

10.

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.9	21.27

$y(218) = ?$

11.

x	1000	1010	1020	1030	1040	1050
y	3	3.0043	3.0086	3.0128	3.0170	3.0212

$y(1044) = ?$

12.

x	1	2	3	4	5
$f(x)$	0	0.3010	0.4771	0.6021	0.6990

$y(5.2) = ?$

13. Given $\sin 20^\circ = 0.3420$, $\sin 25^\circ = 0.4226$, $\sin 30^\circ = 0.5000$, $\sin 35^\circ = 0.5736$, $\sin 40^\circ = 0.6428$. Find $\sin 24^\circ$ and $\sin 42^\circ$ using appropriate interpolation formulae.

14. Fit an interpolating polynomial u_x satisfying

$$u_{-4} = -3, \quad u_{-2} = 5, \quad u_0 = 13, \quad u_2 = 69, \quad u_4 = 221$$

Hence find u_3 and u_6.

15. From the following data estimate the number of students who have scored less than 70 marks.

Marks	0–20	20–40	40–60	60–80	80–100
No. of students	41	62	65	50	17

16. From the following data estimate the number of students scoring marks more than 40 but less than 55.

Marks	30–40	40–50	50–60	60–70	70–80
No. of students	31	42	51	35	31

17. Apply Newton's forward interpolation formula to find $y(0.66)$ given $y(0) = 0, y(0.4) = 0.073, y(0.6) = 0.102, y(0.8) = 0.128, y(1.0) = 0.151$

18. Given that $\sqrt{12} = 3.464$, $\sqrt{14} = 3.742$, $\sqrt{16} = 4$, $\sqrt{18} = 4.243$, $\sqrt{20} = 4.472$, compute $\sqrt{16.5}$ by using Newton's forward interpolation formula.

19. Use Newton's backward interpolation formula to compute u_{25} given $u_{20} = 0.3420, u_{24} = 0.4067, \quad u_{28} = 0.4695, \quad u_{32} = 0.5299$

20. Given the data :

x	310	320	330	340	350	360
$\log_{10} x$	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

compute $\log_{10} 335$ by applying Newton's backward interpolation formula.

ANSWERS

1. $\tan^{-1}\left[\dfrac{h}{1+hx+x^2}\right]$

2. $-4\sin^2 h\cos(2x+2h)$

3. $(e^h-1)^n e^x$

4. $\dfrac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$

5. $\dfrac{\sin a}{\cos ax \ \cos a(x+1)}$

9. 6.354

10. 15.99

11. 3.0186

12. 0.716

13. 0.4068,0.6693

14. $x^3+6x^2+12x+13$; 130, 517

15. 196

16. 69

17. 0.1101

18. 4.062

19. 0.4226

20. 2.525

6.5 Interpolation formulae for unequal intervals

6.51 Divided differences

Let $f(x_0)$, $f(x_1)$, $f(x_2)$, . . . $f(x_n)$ be the values of an unknown function $y = f(x)$ corresponding to the values of $x : x_0, x_1, x_2, \ldots x_n$ at unequal intervals. The *first order divided differences* are defined as follows.

$$f(x_0, x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0}, \quad f(x_1, x_2) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$f(x_2, x_3) = \frac{f(x_3)-f(x_2)}{x_3-x_2} \ \ldots f(x_{n-1}, x_n) = \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$$

The *second order divided differences* are defined as follows.

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2)-f(x_0, x_1)}{x_2-x_0}$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3)-f(x_1, x_2)}{x_3-x_1} \quad \text{etc.,}$$

$$f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1}, x_n)-f(x_{n-2}, x_{n-1})}{x_n-x_{n-2}}.$$

Similarly the other higher order divided differences are defined. The tabular arrangement of these values is called the **divided difference table** and is as follows.

x	$f(x)$	1^{st} D.D	2^{nd} D.D	$\cdots$	n^{th} D.D
x_0	$f(x_0)$				
		$f(x_0, x_1)$			
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$		
		$f(x_1, x_2)$			
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$		
		$f(x_2, x_3)$			
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$f(x_0, x_1, x_2, \cdots x_n)$
x_{n-2}	$f(x_{n-2})$		$f(x_{n-3}, x_{n-2}, x_{n-1})$		
		$f(x_{n-2}, x_{n-1})$			
x_{n-1}	$f(x_{n-1})$		$f(x_{n-2}, x_{n-1}, x_n)$		
		$f(x_{n-1}, x_n)$			
x_n	$f(x_n)$				

Note : *The notation for divided differences should not be confused with the notation for functions of two or more variables.* $f(x_1, x_2)$ *does not stand for a function of two variables* x_1 *and* x_2. *Similarly* $f(x_1, x_2, x_3)$ *is not a function of* x_1, x_2, x_3 *and so on. Here* $x_1, x_2, \cdots$ *are called 'arguments'.*

6.52 Newton's divided difference formula or Newton's general interpolation formula

Statement : *If* $f(x_0), f(x_1), f(x_2), \ldots f(x_n)$ *be a set of values of an unknown function* $f(x)$ *corresponding to the values of* $x : x_0, x_1, x_2, \cdots, x_n$ *at unequal intervals, then*

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \cdots$$
$$+ (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f(x_0, x_1, x_2, \cdots x_n)$$

WORKED PROBLEMS

26. *Use Newton's divided difference formula to find* $f(4)$ *given the data :*

x	0	2	3	6
$f(x)$	-4	2	14	158

>> We shall first form the table of divided differences.

x	$f(x)$	1^{st} D.D	2^{nd} D.D)	3^{rd} D.D
$x_0 = 0$	$f(x_0) = -4$			
		$f(x_0, x_1)$ $\frac{2-(-4)}{2-0} = 3$		
$x_1 = 2$	$f(x_1) = 2$		$f(x_0, x_1, x_2)$ $\frac{12-3}{3-0} = 3$	
		$f(x_1, x_2)$ $\frac{14-2}{3-2} = 12$		$f(x_0, x_1, x_2, x_3)$ $\frac{9-3}{6-0} = 1$
$x_2 = 3$	$f(x_2) = 14$		$f(x_1, x_2, x_3)$ $\frac{48-12}{6-2} = 9$	
		$f(x_2, x_3)$ $\frac{158-14}{6-3} = 48$		
$x_3 = 6$	$f(x_3) = 158$			

We have Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

$$\therefore \quad f(4) = -4 + (4-0)3 + (4-0)(4-2)3 + (4-0)(4-2)(4-3)1$$
$$= -4 + 12 + 24 + 8 = 40$$

Thus $f(4) = \mathbf{40}$

---

27. *Construct the interpolation polynomial for the data given below using Newton's general interpolation formula for divided differences*

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

\>> The divided difference table is as follows.

x	$y = f(x)$	1^{st} D.D	2^{nd} D.D	3^{rd} D.D
$x_0 = 2$	$f(x_0) = 10$			
		$f(x_0, x_1)$ $\frac{96-10}{4-2} = 43$		
$x_1 = 4$	$f(x_1) = 96$		$f(x_0, x_1, x_2)$ $\frac{100-43}{5-2} = 19$	
		$f(x, x_2)$ $\frac{196-96}{5-4} = 100$		$f(x_0, x_1, x_2, x_3)$ $\frac{27-19}{6-2} = 2$
$x_2 = 5$	$f(x_2) = 196$		$f(x_1, x_2, x_3)$ $\frac{154-100}{6-4} = 27$	
		$f(x_2, x_3)$ $\frac{350-196}{6-5} = 154$		$f(x_1, x_2, x_3, x_4)$ $\frac{35-27}{8-4} = 2$
$x_3 = 6$	$f(x_3) = 350$		$f(x_2, x_3, x_4)$ $\frac{259-154}{8-5} = 35$	
		$f(x_3, x_4)$ $\frac{868-350}{8-6} = 259$		$f(x_2, x_3, x_4, x_5)$ $\frac{45-35}{10-5} = 2$
$x_4 = 8$	$f(x_4) = 868$		$f(x_3, x_4, x_5)$ $\frac{439-259}{10-6} = 45$	
		$f(x_4, x_5)$ $\frac{1746-868}{10-8} = 439$		
$x_5 = 10$	$f(x_5) = 1746$			

The fourth order differences are zero as third order differences are same.

We have Newton's general interpolation formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \dots$$

Now, $f(x) = 10 + (x-2)43 + (x-2)(x-4)19 + (x-2)(x-4)(x-5)2$

$$= 10 + (x-2)[43 + (19x - 76) + (x^2 - 9x + 20)2]$$

$$= 10 + (x-2)(2x^2 + x + 7) = 2x^3 - 3x^2 + 5x - 4.$$

Thus the required interpolating polynomial is given by

$$f(x) = 2x^3 - 3x^2 + 5x - 4$$

---

28. *Fit an interpolating polynomial for the data*

$$u_{10} = 355\,,\ u_0 = -5\,,\ u_8 = -21\,,\ u_1 = -14\,,\ u_4 = -125$$

by using Newton's general interpolation formula and hence find u_2

>> We shall arrange the data taking the values of x in the ascending order along with the corresponding values of u_x just for convenience.

(*However this arrangement is not necessary*)

The divided difference table is formed first.

x	$u_x = f(x)$	1^{st} D.D	2^{nd} D.D	3^{rd} D.D
$x_0 = 0$	$f(x_0) = -5$			
		$f(x_0, x_1)$ $\dfrac{-14-(-5)}{1-0} = -9$		
$x_1 = 1$	$f(x_1) = -14$		$f(x_0, x_1, x_2)$ $\dfrac{-37-(-9)}{4-0} = -7$	
		$f(x_1, x_2)$ $\dfrac{-125-(-14)}{4-1} = -37$		$f(x_0, x_1, x_2, x_3)$ $\dfrac{9-(-7)}{8-0} = 2$
$x_2 = 4$	$f(x_2) = -125$		$f(x_1, x_2, x_3)$ $\dfrac{26-(-37)}{8-1} = 9$	
		$f(x_2, x_3)$ $\dfrac{-21+125}{8-4} = 26$		$f(x_1, x_2, x_3, x_4)$ $\dfrac{27-9}{10-1} = 2$
$x_3 = 8$	$f(x_3) = -21$		$f(x_2, x_3, x_4)$ $\dfrac{188-26}{10-4} = 27$	
		$f(x_3, x_4)$ $\dfrac{355-(-21)}{10-8} = 188$		
$x_4 = 10$	$f(x_4) = 355$			

The fourth order differences are zero since the third order differences are same.
We have Newton's general interpolation formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \ldots$$

Now, $f(x) = -5 + (x)(-9) + (x)(x-1)(-7) + (x)(x-1)(x-4)(2)$

$$= -5 + x\left[-9 + (-7x + 7) + (x^2 - 5x + 4)\,2\right]$$
$$= -5 + x\left[2x^2 - 17x + 6\right].$$

Thus $f(x) = \mathbf{u_x = 2x^3 - 17x^2 + 6x - 5}$

Now $f(2) = \mathbf{u_2} = 2\,(2)^3 - 17(2)^2 + 6(2) - 5 = \mathbf{-45}$

---

29. *Given* $u_{20} = 24.37$, $u_{22} = 49.28$, $u_{29} = 162.86$ *and* $u_{32} = 240.5$ *find* u_{28} *by Newton's divided difference formula.*

>> The divided difference table is as follows.

x	$u_x = f(x)$	1^{st} D.D	2^{nd} D.D	3^{rd} D.D
$x_0 = 20$	$f(x_0) = 24.37$			
		$f(x_0, x_1)$ $\dfrac{49.28 - 24.37}{22 - 20}$ $= 12.455$		
$x_1 = 22$	$f(x_1) = 49.28$		$f(x_0, x_1, x_2)$ $\dfrac{16.226 - 12.455}{29 - 20}$ $= 0.419$	
		$f(x_1, x_2)$ $\dfrac{162.86 - 49.28}{29 - 22}$ $= 16.226$		$f(x_0, x_1, x_2, x_3)$ $\dfrac{0.965 - 0.419}{32 - 20}$ $= 0.0455$
$x_2 = 29$	$f(x_2) = 162.86$		$f(x_1, x_2, x_3)$ $\dfrac{25.88 - 16.226}{32 - 22}$ $= 0.965$	
		$f(x_2, x_3)$ $\dfrac{240.5 - 162.86}{32 - 29}$ $= 25.88$		
$x_3 = 32$	$f(x_3) = 240.5$			

We have Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \ldots$$

$\therefore \quad f(28) = 24.37 + (28 - 20)\, 12.455 + (28 - 20)(28 - 22)\, 0.419$
$$+ (28 - 20)(28 - 22)(28 - 29)(0.0455)$$

Thus $\quad f(28) = u_{28} = \mathbf{141.938 \approx 141.94}$

---

30. *Find the cubic polynomial which passes through the points $(2, 4)(4, 56)(9, 711)(10, 980)$ and hence estimate the dependent variable corresponding to the values of the independent variable 3,5,7,11. Also express the interpolating polynomial in powers of $(x - 1)$ and hence estimate the value of $f(x)$ at $x = 1{\cdot}1$ and $1{\cdot}5$*

>> We have to find $f(x)$ where we have by data $f(2) = 4$, $f(4) = 56$, $f(9) = 711$, $f(10) = 980$. The divided difference table is as follows.

x	$f(x)$	1^{st} D.D	2^{nd} D.D	3^{rd} D.D
$x_0 = 2$	$f(x_0) = 4$			
		$f(x_0, x_1)$ $\dfrac{56-4}{4-2} = 26$		
$x_1 = 4$	$f(x_1) = 56$		$f(x_0, x_1, x_2)$ $\dfrac{131-26}{9-2} = 15$	
		$f(x_1, x_2)$ $\dfrac{711-56}{9-4} = 131$		$f(x_0, x_1, x_2, x_3)$ $\dfrac{23-15}{10-2} = 1$
$x_2 = 9$	$f(x_2) = 711$		$f(x_1, x_2, x_3)$ $\dfrac{269-131}{10-4} = 23$	
		$f(x_2, x_3)$ $\dfrac{980-711}{10-9} = 269$		
$x_3 = 10$	$f(x_3) = 980$			

We have Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \dots$$

Now, $f(x) = 4 + (x-2)\,26 + (x-2)(x-4)\,15 + (x-2)(x-4)(x-9)\,1$

$$= 4 + (x-2)\left[26 + (15x - 60) + (x^2 - 13x + 36)\right]$$

$$= 4 + (x-2)\left[x^2 + 2x + 2\right]$$

Thus $\mathbf{f(x) = x^3 - 2x}$ is the required polynomial.

Now $f(3) = 3^3 - 2(3) = \mathbf{21}$; $f(5) = 5^3 - 2(5) = \mathbf{115}$

$f(7) = 7^3 - 2(7) = \mathbf{329}$; $f(11) = 11^3 - 2(11) = \mathbf{1309}$.

We shall now express the polynomial $x^3 - 2x$ in powers of $(x-1)$

Consider $f(x) = x^3 - 2x$

$$= [(x-1)^3 + 3x^2 - 3x + 1] - 2x$$

$$= (x-1)^3 + 3x^2 - 5x + 1$$

$$= (x-1)^3 + 3[(x-1)^2 + 2x - 1] - 5x + 1$$

$$= (x-1)^3 + 3(x-1)^2 + x - 2$$

$$= (x-1)^3 + 3(x-1)^2 + [(x-1) + 1] - 2$$

Thus $\mathbf{f(x) = (x-1)^3 + 3(x-1)^2 + (x-1) - 1}$

Putting $x = 1.1$, $f(1.1) = (1.1-1)^3 + 3(1.1-1)^2 + (1.1-1) - 1 = \mathbf{-0.869}$

Putting $x = 1.5$, $f(1.5) = (1.5-1)^3 + 3(1.5-1)^2 + (1.5-1) - 1 = \mathbf{0.375}$

---

31. *Using Newton's divided difference formula find $f(8)$, $f(15)$ from the following data.*

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

>> We have Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

The divided difference table is formed as follows.

x	$f(x)$	1^{st} D.D	2^{nd} D.D	3^{rd} D.D	4^{th} D.D
$x_0 = 4$	$f(x_0) = 48$				
		$f(x_0, x_1)$ $\frac{100-48}{5-4} = 52$			
$x_1 = 5$	$f(x_1) = 100$		$f(x_0, x_1, x_2)$ $\frac{97-52}{7-4} = 15$		
		$f(x_1, x_2)$ $\frac{294-100}{7-5} = 97$		$f(x_0, x_1, x_2, x_3)$ $\frac{21-15}{10-4} = 1$	
$x_2 = 7$	$f(x_2) = 294$		$f(x_1, x_2, x_3)$ $\frac{202-97}{10-5} = 21$		0
		$f(x_2, x_3)$ $\frac{900-294}{10-7} = 202$		$f(x_1, x_2, x_3, x_4)$ $\frac{27-21}{11-5} = 1$	
$x_3 = 10$	$f(x_3) = 900$		$f(x_2, x_3, x_4)$ $\frac{310-202}{11-7} = 27$		0
		$f(x_3, x_4)$ $\frac{1210-900}{11-10} = 310$		$f(x_2, x_3, x_4, x_5)$ $\frac{33-27}{13-7} = 1$	
$x_4 = 11$	$f(x_4) = 1210$		$f(x_3, x_4, x_5)$ $\frac{409-310}{13-10} = 33$		
		$f(x_4, x_5)$ $\frac{2028-1210}{13-11} = 409$			
$x_5 = 13$	$f(x_5) = 2028$				

Since two values are to be found, we prefer to find the interpolating polynomial $f(x)$.

(Formula can be used twice by taking $x = 8$ and $x = 15$ separately)

$$f(x) = 48 + (x-4)52 + (x-4)(x-5)15 + (x-4)(x-5)(x-7)1$$

$$f(x) = x^3 - x^2, \text{ on simplification.}$$

$$\therefore \quad f(8) = 8^3 - 8^2 = \mathbf{448} \; ; \; f(15) = 15^3 - 15^2 = \mathbf{3150}$$

---

32. *Determine $f(x)$ as a polynomial in x for the following data using Newton's divided difference formula.*

x	-4	-1	0	2	5
y	1245	33	5	9	1335

>> The divided difference table is formed first.

x	$y = f(x)$	1^{st} D.D	2^{nd} D.D	3^{rd} D.D	4^{th} D.D
$x_0 = -4$	$f(x_0) = 1245$				
		$f(x_0, x_1)$ $\frac{33 - 1245}{-1+4} = -404$			
$x_1 = -1$	$f(x_1) = 33$		$f(x_0, x_1, x_2)$ $\frac{-28+404}{0+4} = 94$		
		$f(x_1, x_2)$ $\frac{5-33}{0+1} = -28$		$f(x_0, x_1, x_2, x_3)$ $\frac{10-94}{2+4} = -14$	
$x_2 = 0$	$f(x_2) = 5$		$f(x_1, x_2, x_3)$ $\frac{2+28}{2+1} = 10$		$f(x_0, x_1, x_2, x_3, x_4)$ $\frac{13+14}{5+4} = 3$
		$f(x_2, x_3)$ $\frac{9-5}{2-0} = 2$		$f(x_1, x_2, x_3, x_4)$ $\frac{88-10}{5+1} = 13$	
$x_3 = 2$	$f(x_3) = 9$		$f(x_2, x_3, x_4)$ $\frac{442-2}{5-0} = 88$		
		$f(x_3, x_4)$ $\frac{1335-9}{5-2} = 442$			
$x_4 = 5$	$f(x_4) = 1335$				

On substituting in the Newton's divided difference formula [*Refer previous problem*] we have

$$y = f(x) = 1245 + (x+4)(-404) + (x+4)(x+1)94$$
$$+ (x+4)(x+1)(x)(-14) + (x+4)(x+1)(x)(x-2)3$$

$$f(x) = 1245 + (x+4)\ [-404 + 94x + 94 - 14x^2 - 14x + 3x^3 - 3x^2 - 6x\ ]$$
$$= 1245 + (x+4)\ [\,3x^3 - 17x^2 + 74x - 310\,]$$
$$= 1245 + 3x^4 - 17x^3 + 74x^2 - 310x + 12x^3 - 68x^2 + 296x - 1240$$

Thus $\boldsymbol{f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5}$

EXERCISES

***Use Newton's divided difference formula to find $f(x)$ at the given value of x* [1 to 5]**

1.

x	40	42	44	45
$f(x)$	43833	46568	49431	50912

$f(43) = ?$

2.

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

$f(5) = ?$

3.

x	300	304	305	307
$f(x)$	2.4771	2.4829	2.4843	2.4871

$f(301) = ?$

4.

x	0	2	3	5	6
$f(x)$	0	6	21	105	186

$f(0.5) = ?$

5.

x	1	2	4	7	12
u_x	576	168	-30	48	378

$u_8 = ?$

Use Newton's general interpolation formula to fit an interpolating polynomial for the following. **[6 to 9]**

6. $u_4 = 48, u_5 = 100, u_6 = 180, u_8 = 448, u_{10} = 900, u_{11} = 1210$

7.

x	-1	0	3	6	7
y	3	-6	39	822	1611

8.

x	0	1	2	5
$f(x)$	2	3	12	147

9. $f(-1) = -8, f(0) = 3, f(2) = 1, f(3) = 12$

10. Use Newton's general interpolation formula to find $f(x)$ as a polynomial in powers of $(x-1)$ when $f(x)$ satisfies $f(0) = 4$, $f(2) = 26$, $f(3) = 58$, $f(4) = 112$, $f(7) = 466$, $f(9) = 922$

ANSWERS

1. 47983.2
2. 194
3. 2.4786
4. 0.375
5. 30
6. $x^2(x-1)$
7. $x^4 - 3x^3 + 5x^2 - 6$
8. $x^3 + x^2 - x + 2$
9. $2x^3 - 6x^2 + 3x + 3$
10. $(x-1)^3 + 5(x-1)^2 + 10(x-1) + 10$

6.53 Lagrange's formula for interpolation and inverse interpolation

Statement: *If $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2), \dots y_n = f(x_n)$ be a set of values of an unknown function $y = f(x)$ corresponding to the values of $x : x_0, x_1, x_2, \dots x_n$ not necessarily at equal intervals then*

$$y = f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)\,y_0}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} + \frac{(x-x_0)(x-x_2)(x-x_3)\dots(x-x_n)\,y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\dots(x-x_n)\,y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)} + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})\,y_n}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}$$

Remarks

1. *The special feature of this formula is that the terms of the formula involve only the values in the variables x and y.*

2. *The values of x also need not be equally spaced, nor need they be in any order.*

3. *We can interchange the role of x and y in the formula and the same is called Lagrange's inverse interpolation formula which helps to find x for a given y. The formula is as follows.*

$$x = \frac{(y-y_1)(y-y_2)\dots(y-y_n)\,x_0}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_n)} + \frac{(y-y_0)(y-y_2)\dots(y-y_n)\,x_1}{(y_1-y_0)(y_1-y_2)\dots(y_1-y_n)}$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)\dots(y-y_n)\,x_2}{(y_2-y_0)(y_2-y_1)\dots(y_2-y_3)\dots(y_2-y_n)} + \dots + \frac{(y-y_0)(y-y_1)\dots(y-y_{n-1})\,x_n}{(y_n-y_0)(y_n-y_1)\dots(y_n-y_{n-1})}$$

WORKED PROBLEMS

33. *Use Lagrange's interpolation formula to find $f(4)$ given*

x	0	2	3	6
$f(x)$	-4	2	14	158

>> Let, $x_0 = 0 \quad x_1 = 2 \quad x_2 = 3 \quad x_3 = 6$; $x = 4$

$y_0 = -4 \quad y_1 = 2 \quad y_2 = 14 \quad y_3 = 158$; $y = ?$

We have Lagrange's interpolation formula for four given values,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\,y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)\,y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\,y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)\,y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Substituting the values we have,

$$f(4) = \frac{(4-2)(4-3)(4-6)(-4)}{(0-2)(0-3)(0-6)} + \frac{(4-0)(4-3)(4-6)2}{(2-0)(2-3)(2-6)} + \frac{(4-0)(4-2)(4-6)14}{(3-0)(3-2)(3-6)} + \frac{(4-0)(4-2)(4-3)158}{(6-0)(6-2)(6-3)}$$

$$= \frac{(2)(1)(-2)(-4)}{(-2)(-3)(-6)} + \frac{(4)(1)(-2)2}{(2)(-1)(-4)} + \frac{(4)(2)(-2)14}{(3)(1)(-3)} + \frac{(4)(2)(1)158}{(6)(4)(3)}$$

$$= \frac{-4}{9} - 2 + \frac{224}{9} + \frac{158}{9} = \frac{382}{9} - \frac{22}{9} = \frac{360}{9} = 40$$

Thus $f(4) = 40$

---

34. *Use Lagrange's interpolation formula to find y at x = 10 given*

x	5	6	9	11
y	12	13	14	16

>> Let, $x_0 = 5 \quad x_1 = 6 \quad x_2 = 9 \quad x_3 = 11$, $x = 10$

$y_0 = 12 \quad y_1 = 13 \quad y_2 = 14 \quad y_3 = 16$, $y = ?$.

We have Lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_3)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$\therefore \quad f(10) = \frac{(4)(1)(-1)12}{(-1)(-4)(-6)} + \frac{(5)(1)(-1)13}{(1)(-3)(-5)} + \frac{(5)(4)(-1)14}{(4)(3)(-2)} + \frac{(5)(4)(1)(16)}{(6)(5)(2)}$$

$$f(10) = 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \frac{44}{3} = 14.666... \approx 14.67$$

Thus **y at $x = 10$ is 14.67**

---

35. *Use Lagrange's interpolation formula to fit a polynomial for the data*

x	0	1	3	4
y	−12	0	6	12

Hence estimate y at x = 2

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>> Let $\left.\begin{array}{llll} x_0 = 0 & x_1 = 1 & x_2 = 3 & x_3 = 4 \\ y_0 = -12 & y_1 = 0 & y_2 = 6 & y_3 = 12 \end{array}\right\} \quad y = f(x) = ?$

We have Lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\,y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)\,y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\,y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)\,y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}.$$

Now $y = f(x) = \dfrac{(x-1)(x-3)(x-4)(-12)}{(-1)(-3)(-4)} + 0$

$$+ \frac{x(x-1)(x-4)6}{(3)(2)(-1)} + \frac{x(x-1)(x-3)12}{(4)(3)(1)}$$

$$f(x) = (x-1)(x-3)(x-4) - x(x-1)(x-4) + x(x-1)(x-3)$$

$$= (x-1)\ [(x^2-7x+12) - (x^2-4x) + (x^2-3x)]$$

$$= (x-1)\ [x^2-6x+12] = x^3-7x^2+18x-12.$$

Thus the required polynomial is $\boldsymbol{f(x) = x^3 - 7x^2 + 18x - 12}$

Now, $f(2) = 2^3 - 7(2)^2 + 18(2) - 12 = 4$

Thus $\boldsymbol{f(2) = 4}$

---

36. *The following table gives the normal weights of babies during first eight months of life*

Age (in months)	0	2	5	8
Weight (in pounds)	6	10	12	16

Estimate the weight of the baby at the age of seven months using Lagrange's interpolation formula.

>> Let $\left.\begin{array}{llll} x_0 = 0 & x_1 = 2 & x_2 = 5 & x_3 = 8 \\ y_0 = 6 & y_1 = 10 & y_2 = 12 & y_3 = 16 \end{array}\right\} \quad \begin{array}{l} x = 7 \\ y = ? \end{array}$

We have Lagrange's interpolation formula,

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}+\frac{(x-x_0)(x-x_2)(x-x_3)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}+\frac{(x-x_0)(x-x_1)(x-x_2)y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$f(7)=\frac{(5)(2)(-1)6}{(-2)(-5)(-8)}+\frac{(7)(2)(-1)10}{(2)(-3)(-6)}+\frac{(7)(5)(-1)12}{(5)(3)(-3)}+\frac{(7)(5)(2)16}{(8)(6)(3)}$$

$$f(7)=\frac{3}{4}-\frac{35}{9}+\frac{28}{3}+\frac{70}{9}=13.97\approx 14.$$

Thus the approx. wt. of the baby at the age of 7 months is 14 pounds.

---

37. *If $y(1)=3$, $y(3)=9$, $y(4)=30$, $y(6)=132$, find Lagrange's interpolation polynomial that takes on these values.*

>> Let $\left.\begin{matrix} x_0=1 & x_1=3 & x_2=4 & x_3=6 \\ y_0=3 & y_1=9 & y_2=30 & y_3=132 \end{matrix}\right\}$ $y=f(x)=?$

We have Lagrange's interpolation formula,

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}+\frac{(x-x_0)(x-x_2)(x-x_3)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}+\frac{(x-x_0)(x-x_1)(x-x_2)y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$f(x)=\frac{(x-3)(x-4)(x-6)3}{(-2)(-3)(-5)}+\frac{(x-1)(x-4)(x-6)9}{(2)(-1)(-3)}$$

$$+\frac{(x-1)(x-3)(x-6)30}{(3)(1)(-2)}+\frac{(x-1)(x-3)(x-4)132}{(5)(3)(2)}$$

$$=\frac{-1}{10}(x-3)(x-4)(x-6)+\frac{3}{2}(x-1)(x-4)(x-6)$$

$$-5(x-1)(x-3)(x-6)+\frac{22}{5}(x-1)(x-3)(x-4)$$

$$=\frac{1}{10}\Big[-(x-3)(x-4)(x-6)+15(x-1)(x-4)(x-6)$$

$$-50(x-1)(x-3)(x-6)+44(x-1)(x-3)(x-4)\Big]$$

We can use the basic expansion formula,

$$(x-a)(x-b)(x-c) = x^3 - x^2(a+b+c) + x(ab+bc+ca) - abc$$

$$f(x) = \frac{1}{10}\Big[-(x^3-13x^2+54x-72) + 15(x^3-11x^2+34x-24)$$

$$-50(x^3-10x^2+27x-18) + 44(x^3-8x^2+19x-12) \Big]$$

$$= \frac{1}{10}\left[8x^3 - 4x^2 - 58x + 84 \right]$$

Thus the required polynomial is $f(x) = \frac{1}{5}\left[4x^3 - 2x^2 - 29x + 42 \right]$

---

38. *Given $u_0 = 707$, $u_2 = 819$, $u_3 = 866$ and $u_6 = 966$ compute u_4 using Lagrange's interpolation formula.*

>> Let $x_0 = 0$ $x_1 = 2$ $x_2 = 3$ $x_3 = 6$; $x = 4$

$y_0 = 707$ $y_1 = 819$ $y_2 = 866$ $y_3 = 966$; $y = ?$

We have Lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\,y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)\,y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\,y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)\,y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$\therefore \quad f(4) = \frac{(2)(1)(-2)\,707}{(-2)(-3)(-6)} + \frac{(4)(1)(-2)\,819}{(2)(-1)(-4)}$$

$$+ \frac{(4)(2)(-2)\,866}{(3)(1)(-3)} + \frac{(4)(2)(1)\,966}{(6)(4)(3)} = 906.43$$

Thus $u_4 = 906.43$

---

39. *Applying Lagrange's formula inversely find x when $y = 6$ given the data*

x	20	30	40
y	2	4.4	7.9

>> Let $x_0 = 20$ $x_1 = 30$ $x_2 = 40$; $x = ?$

$y_0 = 2$ $y_1 = 4.4$ $y_2 = 7.9$; $y = 6$

We have Lagrange's inverse interpolation formula,

$$x = \frac{(y-y_1)(y-y_2)x_0}{(y_0-y_1)(y_0-y_2)} + \frac{(y-y_0)(y-y_2)x_1}{(y_1-y_0)(y_1-y_2)} + \frac{(y-y_0)(y-y_1)x_2}{(y_2-y_0)(y_2-y_1)}$$

$$x(6) = \frac{(1.6)(-1.9)20}{(-2.4)(-5.9)} + \frac{(4)(-1.9)30}{(2.4)(-3.5)} + \frac{(4)(1.6)40}{(5.9)(3.5)}$$

$$x(6) = 35.2462$$

Thus the value of x when $y = 6$ is 35.2462

---

40. *The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9, 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable ?*

>> For the function $y = f(x)$, x is the independent variable and y is the dependent variable.

Let $x_0 = 3 \quad x_1 = 7 \quad x_2 = 9 \quad x_3 = 10$; $\quad y_0 = 168 \quad y_1 = 120 \quad y_2 = 72 \quad y_3 = 63$; $\quad x = 6,\ y = ?$

We have Lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_3)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$\therefore \quad f(6) = \frac{(-1)(-3)(-4)168}{(-4)(-6)(-7)} + \frac{(3)(-3)(-4)120}{(4)(-2)(-3)} + \frac{(3)(-1)(-4)72}{(6)(2)(-1)} + \frac{(3)(-1)(-3)63}{(7)(3)(1)}$$

$$= 12 + 180 - 72 + 27 = 147$$

Thus the estimate at position 6 of the independent variable is 147

---

41. *Apply Lagrange's formula inversely to find a root of the equation $f(x) = 0$ given that $f(30) = -30$, $f(34) = -13$ $f(38) = 3$, $f(42) = 18$.*

>> Here we have to find x such that $f(x) = y = 0$

Let $x_0 = 30 \quad x_1 = 34 \quad x_2 = 38 \quad x_3 = 42$; $\quad y_0 = -30 \quad y_1 = -13 \quad y_2 = 3 \quad y_3 = 18$; $\quad x = ?\ y = 0$

We have Lagrange's inverse interpolation formula,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)x_0}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} + \frac{(y-y_0)(y-y_2)(y-y_3)x_1}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)x_2}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} + \frac{(y-y_0)(y-y_1)(y-y_2)x_3}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}$$

$$x(0) = \frac{(13)(-3)(-18)30}{(-17)(-33)(-48)} + \frac{(30)(-3)(-18)(34)}{(17)(-16)(-31)}$$

$$+ \frac{(30)(13)(-18)38}{(33)(16)(-15)} + \frac{(30)(13)(-3)42}{(48)(31)(15)}$$

$$= -0.7821 + 6.5322 + 33.6818 - 2.2016$$

$\therefore \quad x(0) = 37.2303$

Thus an approximate root of $f(x) = 0$ is 37.2303

---

42. *Fit an interpolating polynomial of the form $x = f(y)$ for the data and hence find $x(5)$, $y(5)$*

x	2	10	17
y	1	3	4

>> Let $\left.\begin{matrix} x_0 = 2 & x_1 = 10 & x_2 = 17 \\ y_0 = 1 & y_1 = 3 & y_2 = 4 \end{matrix}\right\} x = f(y) = ?$

Since the polynomial in y is needed, we consider Lagrange's inverse interpolation formula,

$$x = f(y) = \frac{(y-y_1)(y-y_2)x_0}{(y_0-y_1)(y_0-y_2)} + \frac{(y-y_0)(y-y_2)x_1}{(y_1-y_0)(y_1-y_2)} + \frac{(y-y_0)(y-y_1)x_2}{(y_2-y_0)(y_2-y_1)}$$

$$= \frac{(y-3)(y-4)2}{(-2)(-3)} + \frac{(y-1)(y-4)10}{(2)(-1)} + \frac{(y-1)(y-3)17}{(3)(1)}$$

$$= \frac{1}{3}(y-3)(y-4) - 5(y-1)(y-4) + \frac{17}{3}(y-1)(y-3)$$

$$x = \frac{1}{3}(y^2 - 7y + 12) - 5(y^2 - 5y + 4) + \frac{17}{3}(y^2 - 4y + 3)$$

Thus $\quad$ **$x = f(y) = y^2 + 1$ is the required polynomial**

(i) At $y = 5$, $x = 5^2 + 1 = 26$

(ii) At $x = 5$, $5 = y^2 + 1$ or $y^2 = 4$ or $y = 2$

Thus **$x(5) = 26$ and $y(5) = 2$**

---

43. *Using Lagrange's formula find the interpolating polynomial that approximates to the function described by the following table.*

x	0	1	2	5
$f(x)$	2	3	12	147

\>> Let $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 5$; $y_0 = 2,\ y_1 = 3,\ y_2 = 12,\ y_3 = 147$ $\quad y = f(x) = ?$

We have Lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\,y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)\,y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\,y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)\,y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$y = f(x) = \frac{(x-1)(x-2)(x-5)\cdot 2}{(-1)(-2)(-5)} + \frac{x(x-2)(x-5)\cdot 3}{(1)(-1)(-4)}$$

$$+ \frac{x(x-1)(x-5)\cdot 12}{(2)(1)(-3)} + \frac{x(x-1)(x-2)\cdot 147}{(5)(4)(3)}$$

$$f(x) = -\frac{1}{5}(x-1)(x-2)(x-5) + \frac{3}{4}x(x-2)(x-5)$$

$$-2x(x-1)(x-5) + \frac{49}{20}x(x-1)(x-2)$$

$$f(x) = \frac{1}{20}\Big[-4(x^3-8x^2+17x-10) + 15(x^3-7x^2+10x)$$

$$-40(x^3-6x^2+5x) + 49(x^3-3x^2+2x)\Big]$$

$$= \frac{1}{20}(20x^3+20x^2-20x+40)$$

Thus $\mathbf{f(x) = x^3+x^2-x+2}$

---

44. *Using Lagrange's interpolation method, find the value of $f(x)$ at $x = 5$ given the values*

x	1	3	4	6
$f(x)$	3	9	30	132

\>> Let $x_0 = 1,\ x_1 = 3,\ x_2 = 4,\ x_3 = 6$; $y_0 = 3,\ y_1 = 9,\ y_2 = 30,\ y_3 = 132$ $\quad x = 5,\ f(5) = ?$

On substituting in the Lagrange's interpolation formula (*Refer the previous problem*) we have

$$f(5) = \frac{(2)\ (1)\ (-1)\ 3}{(-2)\ (-3)\ (-5)} + \frac{(4)\ (1)\ (-1)\ 9}{(2)\ (-1)\ (-3)} + \frac{(4)\ (2)\ (-1)\ 30}{(3)\ (1)\ (-2)} + \frac{(4)\ (2)\ (1)\ 132}{(5)\ (3)\ (2)}$$

$$f(5) = \frac{1}{5} - 6 + 40 + \frac{176}{5} = \frac{347}{5} = 69.4$$

Thus **$f(5) = 69.4$**

---

45. *Using Lagrange's interpolation formula, find f (9) from the following data.*

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Let, $x_0 = 5,\quad x_1 = 7,\quad x_2 = 11,\quad x_3 = 13,\quad x_4 = 17$

$y_0 = 150,\ y_1 = 392,\ y_2 = 1452,\ y_3 = 2366,\ y_4 = 5202$

We have Lagrange's interpolation formula for a set of 5 given values,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)\,y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)\,y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)\,y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)\,y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\,y_4}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

Substituting the values we have,

$$f(9) = \frac{(2)(-2)(-4)(-8)\cdot 150}{(-2)(-6)(-8)(-12)} + \frac{(4)(-2)(-4)(-8)\cdot 392}{(2)(-4)(-6)(-10)}$$

$$+ \frac{(4)(2)(-4)(-8)\cdot 1452}{(6)(4)(-2)(-6)} + \frac{(4)(2)(-2)(-8)\cdot 2366}{(8)(6)(2)(-4)} + \frac{(4)(2)(-2)(-4)\cdot 5202}{(12)(10)(6)(4)}$$

Thus $f(9)$ = **810**, on simplification.

Note : *When more number of values are given at unequal intervals for interpolation without the specific mention of the formula, we must prefer Newton's general interpolation formula.*

EXERCISES

Apply Lagrange's formula to find y at the given value of x

1.

x	2	5	8	14
y	94.8	87.9	81.3	68.7

$y(11) = ?$

2.

x	1.2	2.0	2.5	3.0
y	1.36	0.58	0.34	0.20

$y(1.6) = ?$

3.

x	10	12	19	22
y	24	48	162	200

$y(18) = ?$ (to the nearest integer)

4. If $f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(7) = 128$, find $f(5)$ and $f(6)$ using Lagrange's interpolation formula.

Use Lagrange's interpolation formula to fit a polynomial for the following data

5.

x	1	2	3	4
$f(x)$	5	19	49	101

6.

x	1	2	4	5
y	14	41	197	350

7. Use Lagrange's inverse interpolation formula to find the value of x for $y = 100$ given $y(3) = 6$, $y(5) = 24$, $y(7) = 58$, $y(9) = 108$, $y(11) = 174$.

8. Use Lagrange's formula in the appropriate form to fit a polynomial of the form $x = f(y)$ for the data & hence find x when $y = 4$

x	0	2	3	5
y	2	10	17	37

ANSWERS

1. 74.925
2. 0.8932
3. 145
4. 38.8, 74
5. $x^3 + 2x^2 + 2x + 1$
6. $2x^3 + 3x^2 + 4x + 5$
7. 8.66
8. $x = y^2 + 2y + 2$; 26

Note : ***We briefly present the topic Numerical Differentiation for the benefit of readers with Illustrative examples.***

Suppose y_0 , y_1 , y_2 , . . . y_n are the values of an unknown function $y = f(x)$ corresponding to $x : x_0$, x_1, x_2, . . . x_n the process of computing $f'(x)$, $f''(x)$, $f'''(x)$, etc., at some particular value of the independent variable x is known as ***Numerical Differentiation.***

The approximate value of these derivatives are obtained by differentiating an appropriate interpolation formula.

Case - (i) : The given values of x are equidistant and the given x is near to x_0.

We prepare the forward difference table and consider Newton's forward interpolation formula :

$$f(x_0 + rh) = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

Differentiating w.r.t r on both sides we get,

$$f'(x_0 + rh)\cdot h = \Delta y_0 + (2r-1)\frac{\Delta^2 y_0}{2!} + (3r^2 - 6r + 2)\frac{\Delta^3 y_0}{3!} + \cdots \qquad \ldots (1)$$

Differentiating w.r.t r again we get,

$$f''(x_0 + rh)\cdot h^2 = \Delta^2 y_0 + (6r-6)\frac{\Delta^3 y_0}{3!} + \cdots$$

i.e., $$f''(x_0 + rh)\cdot h^2 = \Delta^2 y_0 + (r-1)\,\Delta^3 y_0 + \cdots \qquad \ldots (2)$$

Case - (ii) : The given values are equidistant and the given x is near to x_n. We prepare the backward difference table and consider Newton's backward interpolation formula :

$$f(x_n + rh) = y_n + r\,\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

Differentiating w.r.t r on both sides we get,

$$f'(x_n + rh)\cdot h = \nabla y_n + (2r+1)\frac{\nabla^2 y_n}{2!} + (3r^2 + 6r + 2)\frac{\nabla^3 y_n}{3!} + \cdots \qquad \ldots (3)$$

Differentiating again w.r.t. r we get,

$$f''(x_n + rh)\cdot h^2 = \nabla^2 y_n + (r+1)\,\nabla^3 y_n + \cdots \qquad \ldots (4)$$

Working procedure for problems

- The given value of x is appropriately equated to $x_0 + rh$ or $x_n + rh$ to obtain the value of r.
- The value of the finite differences along with the value of r is substituted into the relevant result [(1) to (4)] obtained on differentiating the interpolation formula .
- The same procedure is adopted for finding third order, fourth order . . . .. derivatives at the given value of the independent variable.

Note : *If the interval is unequal then also we can find the derivatives at the given value of the independent variable by differentiating Newton's general interpolation formula for* $y = f(x)$.

ILLUSTRATIVE EXAMPLES

1. *For the following data find* $f'(1)$ *and* $f''(3)$. *Verify the answer by fitting an interpolating polynomial.*

x	0	2	4	6	8
$f(x)$	7	13	43	145	367

>> The forward difference table is as follows.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
$x_0 = 0$	$y_0 = 7$				
		$\Delta y_0 = 6$			
2	13		$\Delta^2 y_0 = 24$		
		30		$\Delta^3 y_0 = 48$	
4	43		72		0
		102		48	
6	145		120		
		222			
8	367				

We have to first obtain results (1) and (2).

To find $f'(1)$ we take $x_0 + rh = 1$ where $x_0 = 0$ and $h = 2$.

Hence $r = 1/2$ and (1) becomes

$$f'(1) \cdot 2 = 6 + 0 + (3/4 - 3 + 2)8 = 4 \qquad \therefore \quad \mathbf{f'(1) = 2}$$

To find $f''(3)$ we take $x_0 + rh = 3$ and we obtain $r = 3/2$

Hence (2) becomes

$$f''(3)\cdot 4 = 24 + (1/2)\,48 = 48 \qquad \therefore \quad f''(3) = \mathbf{12}$$

Verification : We have $r = \dfrac{x - x_0}{h} = \dfrac{x}{2}$

We shall find the interpolating polynomial from the Newton's forward interpolation formula.

$$f(x_0 + rh) \text{ or } f(x) = 7 + \frac{x}{2}(6) + \frac{x}{2}\left(\frac{x}{2} - 1\right)12 + \frac{x}{2}\left(\frac{x}{2} - 1\right)\left(\frac{x}{2} - 2\right)8$$

i.e., $\quad f(x) = 7 + 3x + x(x-2)3 + x(x-2)(x-4)$

$\therefore \quad f(x) = x^3 - 3x^2 + 5x + 7$

Differentiating this successively w.r.t. x we obtain

$$f'(x) = 3x^2 - 6x + 5 \text{ and } f''(x) = 6x - 6$$

Now $\quad f'(1) = 3 - 6 + 5 = \mathbf{2}\ ; \quad f''(3) = 18 - 6 = \mathbf{12}$

Thus the results are verified.

---

2. *Given the data*

x	-2	-1	0	1	2	3
y	0	0	6	24	60	120

Compute $\left(\dfrac{dy}{dx}\right)_{x=2}$ *and* $\left(\dfrac{d^2y}{dx^2}\right)_{x=4.5}$

>> The values of x are near to the last value 3 and we first prepare the backward difference table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
-2	0				
		0			
-1	0		6		
		6		6	
0	6		12		0
		18		6	
1	24		18		0
		36		$\nabla^3 y_n = 6$	
2	60		$\nabla^2 y_n = 24$		
		$\nabla y_n = 60$			
$x_n = 3$	$y_n = 120$				

We have to first obtain results (3) and (4).

We need to find $f'(2)$ and $f''(4.5)$.

Accordingly we have to take $x_n + rh = 2$ and $x_n + rh = 4.5$ respectively.

We obtain $r = -1$ and $r = 1.5$ respectively since $x_n = 3,\ h = 1$.

Hence (3) becomes

$$f'(2) \cdot 1 = 60 + (-1)(12) + (-1)(1) = 47$$

Also (4) becomes

$$f''(4.5) \cdot 1 = 24 + (2.5)\,6 = 39$$

Thus $\mathbf{f'(2) = 47}$ and $\mathbf{f''(4.5) = 39}$

---

3. *Compute* $\dfrac{dy}{dx}$ *at* $x = 3$ *and* $\dfrac{d^2y}{dx^2}$ *at* $x = 7$ *for the data in Problem - 27.*

>> The values of x are not equidistant and we have to compute $f'(3)$ and $f''(7)$.

Disfferentiating the expression of $f(x) = y$ w.r.t. x the Newton's general interpolation formula we have

$$f'(x) = f(x_0, x_1) + [(x - x_0) + (x - x_1)]\, f(x_0, x_1, x_2)$$

$$+ [(x - x_0)(x - x_1) + (x - x_1)(x - x_2) + (x - x_2)(x - x_0)]\, f(x_0, x_1, x_2, x_3) + \cdots \qquad \ldots (1)$$

Again differentiating (1) w.r.t x, we have

$$f''(x) = 2 f(x_0, x_1, x_2) + 2\left[(x - x_0) + (x - x_1) + (x - x_2)\right] f(x_0, x_1, x_2, x_3) \qquad \ldots (2)$$

Using the values of the divided differences as in the divided difference table of Problem - 27 we compute $f'(3)$ from (1) and $f''(7)$ from (2) as follows.

$$f'(3) = 43 + [1 + (-1)]\, 19 + [(1)(-1) + (-1)(-2) + (-2)(1)]\, 2$$

$\therefore$ $\quad \mathbf{f'(3)} = 43 + 0 - 2 = \mathbf{41}$

$$f''(7) = 2(19) + 2[5 + 3 + 2]\, 2$$

$\therefore$ $\quad f''(7) = 38 + 40 = \mathbf{78}$

Note : ***Verification***

We have obtained in Problem - 27, $f(x) = 2x^3 - 3x^2 + 5x - 4$

$\therefore \quad f'(x) = 6x^2 - 6x + 5$ and $f''(x) = 12x - 6$

From these, $f'(3) = 54 - 18 + 5 = \mathbf{41}$ and $f''(7) = 84 - 6 = \mathbf{78}$

6.6 Numerical Integration

This is the process of obtaining approximately the value of the definite integral $I = \int_a^b y\,dx$ without actually integrating the function but only using the values of y at some points of x equally spaced over $[a, b]$. We need techniques to accomplish this because, not all functions can be integrated by the various standard methods of integration. Further there are many situations where we have only some values of y corresponding to equidistant values of x.

We present three rules/formulae to obtain the value of the definite integral $I = \int_a^b y\,dx$ ***numerically.*** *The following is a common step for applying any of the rule.*

The interval $[a, b]$ is divided into n equal parts of width h where $h = (b-a)/n$.

Let $a = x_0$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \cdots x_n = x_0 + nh = b$ be the points of division. Also let $y_0 = f(x_0)$, $y_1 = f(x_1), \cdots y_n = f(x_n)$ be the corresponding values of $f(x)$.

Now we have a set of values of $y = f(x)$ at equidistant points of x and the values (x, y) are tabulated.

x	$x_0 = a$	x_1	x_2	x_3	$\cdots$	$x_n = b$
y	y_0	y_1	y_2	y_3	$\cdots$	y_n

The rules are as follows :

6.61 Simpson's one third rule

$$I = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]$$

6.62 Simpson's three eighth rule

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})\right]$$

6.63 Weddle's rule

$$I = \frac{3h}{10} \sum_{i=0}^{n} k\, y_i \quad \text{where, } k = 1, 5, 1, 6, 1, 5, 2, 5, 1, 6, 1, 5, 2, \cdots$$

However it should be noted that when $n = 6$ we have Weddle's rule,

$$I = \int_{x_0}^{x_0+6h} y\, dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 \right]$$

Notes : *(i) When we divide the interval* $[a, b]$ *into n equal parts there will be* $(n+1)$ *values of* $x : a = x_0, x_1, x_2, \ldots, x_n = b$. *The corresponding values of* y, *also* $(n+1)$ *in number are referred to as the 'ordinates'. So we can conclude that if there are* $(n+1)$ *ordinates there must be n equal divisions.*

(ii) ***It is very important to know that***

(a) to apply Simpson's 1/3 rd rule n must be a multiple of 2
(b) to apply Simpson's 3/8 th rule n must be a multiple of 3
(c) to apply Weddle's rule n must be a multiple of 6

When $n = 6$ *or multiple of 6 all the rules can be applied to find the approximate value of the given definite integral.*

Working procedure for problems

- Given the definite integral $I = \int_a^b y\, dx$ for evaluation, first divide the interval $[a, b]$ into appropriate number of equal parts (*strips*) so as to apply the desired rule. $a = x_0, x_1, x_2, \ldots x_n = b$ be the points of division inclusive of the ends.
- Prepare a table consisting the values of x and the corresponding computed values of y denoted respectively by $y_0, y_1, y_2, \ldots, y_n$
- Substitute values from this table into the appropriate rule to obtain the approximate value of the given definite integral.

Note : *Sometimes it is possible to deduce the value of a certian quantity by equating the theoretical value of the definite integral (when exists) with that of the numerical value obtained without integration using the rule.*

WORKED PROBLEMS

46. *Evaluate* $\int_0^6 3x^2\, dx$ *dividing the interval* $[0, 6]$ *into six equal parts by applying*

(a) Simpson's 1/3 rd rule (b) Simpson's 3/8 th rule (c) Weddle's rule

>> [*We have taken a very simple problem with the intension to get a comparison with the theoretical answer as the given function is easily integrable.* $\int_0^6 3x^2\, dx = \left[x^3\right]_0^6 = 216 \cdots$

Now let us work the problem by numerical method applying various rules without the involvement of integration.

Dividing $[0, 6]$ into 6 equal parts $(n = 6)$ the length of each part is $\frac{6-0}{6} = 1 = h$.

The points of division are got by starting with the left end point of the interval which being 0 and keep on adding $h = 1$ to it so as to reach the right end point of the interval 6.

The points of division are $x = 0, 1, 2, 3, 4, 5, 6$ and we can easily find the corresponding values of the given function $y = 3x^2$.

The set of values of x and y are represented in the following table.

x	0	1	2	3	4	5	6
$y = 3x^2$	0	3	12	27	48	75	108
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(a) *By Simpson's 1/3 rd rule*

$$\int_a^b y\, dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \ldots + y_{n-2})\right]$$

Here $n = 6$, $h = 1$, $a = 0$, $b = 6$ and $y = 3x^2$

$$\therefore \quad \int_0^6 3x^2\, dx = \frac{1}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$ie., \quad \int_0^6 3x^2\, dx = \frac{1}{3}\left[(0 + 108) + 4(3 + 27 + 75) + 2(12 + 48)\right] = \mathbf{216}$$

■

(b) *By Simpson's 3/8 th rule*

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\ldots+y_{n-1})+2(y_3+y_6+\ldots+y_{n-3})\right]$$

Here $n = 6$, $h = 1$

$$\therefore \quad \int_0^6 3x^2\,dx = \frac{3}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$ie., \quad \int_0^6 3x^3\,dx = \frac{3}{8}\left[(0+108)+3(3+12+48+75)+2(27)\right] = \mathbf{216}$$

(c) *By Weddle's rule*

$$I = \int_{x_0}^{x_0+6h} y\,dx = \frac{3h}{10}(y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6)$$

$$\therefore \quad \int_0^6 3x^2\,dx = \frac{3}{10}(0+15+12+162+48+375+108) = \mathbf{216}$$

Thus $\displaystyle\int_0^6 \mathbf{3x^2\,dx = 216}$, from all the three rules.

---

47. *Evaluate* $\displaystyle\int_0^1 \frac{dx}{1+x^2}$ *by using Simpson's 1/3 rd rule taking four equal strips and hence deduce an approximate value of* π.

>> Let us divide [0,1] into 4 equal strips ($n = 4$)

$\therefore$ length of each strip : $h = \dfrac{1-0}{4} = \dfrac{1}{4}$

The points of division are $x = 0, \dfrac{1}{4}, \dfrac{2}{4} = \dfrac{1}{2}, \dfrac{3}{4}, \dfrac{4}{4} = 1$

By data $y = \dfrac{1}{1+x^2}$

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$\therefore$ At $x = 0$, $y = \dfrac{1}{1+0^2} = 1$

$x = \dfrac{1}{4}$, $y = \dfrac{1}{1 + (1/4)^2} = \dfrac{16}{17}$

$x = \dfrac{1}{2}$, $y = \dfrac{1}{1 + (1/2)^2} = \dfrac{4}{5}$

$x = \dfrac{3}{4}$, $y = \dfrac{1}{1 + (3/4)^2} = \dfrac{16}{25}$

$x = 1$, $y = \dfrac{1}{1+(1)^2} = \dfrac{1}{2}$

Now we have the following table.

x	0	1/4	1/2	3/4	1
$y = 1/1+x^2$	1	16/17	4/5	16/25	1/2
	y_0	y_1	y_2	y_3	y_4

Simpson's 1/3 rd rule for $n = 4$ is given by

$$\int_a^b y\,dx = \frac{h}{3}\Big[(y_0+y_4)+4(y_1+y_3)+2y_2\Big]$$

$$\therefore \quad \int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/4}{3}\left[\left(1+\frac{1}{2}\right)+4\left(\frac{16}{17}+\frac{16}{25}\right)+2\cdot\frac{4}{5}\right] = 0.7854$$

Thus $\displaystyle\int_0^1 \frac{dx}{1+x^2}$ **= 0.7854**

To deduce the value of π : We perform theoretical integration and equate the resulting value to the numerical value obtained.

$$\int_0^1 \frac{dx}{1+x^2} = \Big[\tan^{-1} x\Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

We must have, $\dfrac{\pi}{4} = 0.7854$ or $\pi = 4(0.7854) = 3.1416$

Thus $\boldsymbol{\pi = 3.1416}$ (*actual value of* $\pi = 3.1415927\ \ldots$)

48. *Evaluate* $\int_0^1 \frac{dx}{1+x}$ *taking seven ordinates by applying Simpson's 3/8 th rule. Hence deduce the value of* $\log_e 2$

>> 7 ordinates means that the given interval $[0, 1]$ must be divided into 6 equal parts. That is $n = 6$.

The length of each part is $\frac{1-0}{6} = \frac{1}{6} = h$

The values of x and $y = 1/1+x$ are tabulated.

x	0	1/6	2/6	3/6	4/6	5/6	6/6
$y = 1/1+x$	1	6/7	3/4	2/3	3/5	6/11	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's 3/8 th rule for $n = 6$ is given by

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$\therefore \quad \int_0^1 \frac{1}{1+x}\,dx = \frac{1}{16}\left[\left(1+\frac{1}{2}\right)+3\left(\frac{6}{7}+\frac{3}{4}+\frac{3}{5}+\frac{6}{11}\right)+2\cdot\frac{2}{3}\right] = 0.6932$$

Thus $\int_0^1 \frac{dx}{1+x} = \mathbf{0.6932}$

To deduce the value of $\log_e 2$:

Integrating the given function theoretically,

$$\int_0^1 \frac{1}{1+x}\,dx = \left[\log_e(1+x)\right]_0^1$$

$$= \log_e 2 - \log_e 1 = \log_e 2, \quad \text{since } \log_e 1 = 0$$

Equating this with the obtained numerical value we have,

$$\mathbf{\log_e 2 = 0.6932}$$

Note : *The actual value of* $\log_e 2 = 0.6931471$

- -

49. *Use Simpson's 3/8 th rule to evaluate* $\int_1^4 e^{1/x}\, dx$

>> To apply Simpson's 3/8th rule, n must be a multiple of 3 and we shall take $n = 3$ itself.

$\therefore$ the length of each part/strip (h) $= \frac{4-1}{3} = 1.$

Simpson's 3/8 th rule for $n = 3$ is given by

$$\int_a^b y\, dx = \frac{3h}{8} [(y_0 + y_3) + 3 (y_1 + y_2)]$$

The table showing the values of x and the corresponding values of $y = e^{1/x}$ correct to four decimal places is as follows.

x	1	2	3	4
$y = e^{1/x}$	2.7183	1.6487	1.3956	1.2840
	y_0	y_1	y_2	y_3

Substituting these values in the rule,

$$\int_1^4 e^{1/x}\, dx = \frac{3}{8}\left[(2.7183 + 1.284) + 3 (1.6487 + 1.3956) \right] = 4.9257$$

Thus $\boldsymbol{\int_1^4 e^{1/x}\, dx = 4.9257}$

Note : $e^{1/x}$ *cannot be integrated theoretically.*

---

50. *Find the approximate value of* $\int_0^{\pi/2} \sqrt{\cos\theta}\; d\theta$ *by Simpson's 1/3 rd rule by dividing* $[0 , \pi/2]$ *into 6 equal parts.*

>> Length of each part (h) $= \frac{\pi/2 - 0}{6} = \frac{\pi}{12}$ or 15^0

The values of θ and the corresponding values of $f(\theta) = \sqrt{\cos\theta}$ correct to four decimal places are tabulated.

θ^o	0^o	15^o	30^o	45^o	60^o	75^o	90^o
$\sqrt{\cos\theta}$	1	0.9828	0.9306	0.8409	0.7071	0.5087	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's 1/3 rd rule for $n = 6$ is given by

$$\int_a^b y\,d\theta = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

Here $h = \pi/12$ where $\pi = 22/7$

$$\therefore \quad \int_0^{\pi/2} \sqrt{\cos\theta}\,d\theta = \frac{\pi/12}{3}\left[(1+0)+4(0.9828+0.8409+0.5087)\right.$$
$$\left.+2(0.9306+0.7071)\right] = 1.1873$$

Thus $\displaystyle\int_0^{\pi/2} \sqrt{\cos\theta}\,d\theta = \mathbf{1.1873}$

---

51. *Evaluate* $\displaystyle\int_0^3 \frac{dx}{(1+x)^2}$ *by Simpson's 3/8 th rule.*

>> Since the number of strips to be taken is not mentioned explicitly, let us take $n = 3$ and apply Simpson's 3/8 th rule.

Length of each strip $(h) = \dfrac{3-0}{3} = 1.$

We prepare the following table.

x	0	1	2	3
$y = 1/(1+x)^2$	1	1/4	1/9	1/16
	y_0	y_1	y_2	y_3

Simpson's 3/8 th rule for $n = 3$ is given by,

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0+y_3)+3(y_1+y_2)\right]$$

$$\therefore \quad \int_0^3 \frac{1}{(1+x)^2}\,dx = \frac{3}{8}\left[\left(1+\frac{1}{16}\right)+3\left(\frac{1}{4}+\frac{1}{9}\right)\right] = 0.8047$$

Thus $\int_0^3 \frac{dx}{(1+x)^2} = \mathbf{0.8047}$

Remark : *Suppose we integrate the function theoretically,*

$$\int_0^3 \frac{1}{(1+x)^2}\, dx = \left. \frac{-1}{1+x} \right|_0^3 = \frac{-1}{4} + 1 = \frac{3}{4} = 0.75$$

Had we taken more strips (lesser h), we would have got the answer closer to the actual value. If we work out by taking $n = 6$ *which gives* $h = 1/2$ *the answer works out to be* 0.7582621 *which is relatively closer to the actual value* 0.75

---

52. *Use Simpson's 1/3 rd rule with seven ordinates to evaluate* $\int_2^8 \frac{dx}{\log_{10} x}$

>> Seven ordinates means that the given interval [2, 8] must be divided into 6 equal parts.($n = 6$)

Length of each part (h) $= \frac{8-2}{6} = 1$

The values of x and $y = 1 / \log_{10} x$ are tabulated.

x	2	3	4	5	6	7	8
$y = 1/\log_{10} x$	3.3219	2.0959	1.661	1.4307	1.2851	1.1833	1.1073
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's 1/3 rd rule for $n = 6$ is given by,

$$\int_a^b y\, dx = \frac{h}{3} \left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right]$$

$$\therefore \quad \int_2^8 \frac{dx}{\log_{10} x} = \frac{1}{3} \left[(3.3219 + 1.1073) + 4(2.0959 + 1.4307 + 1.1833) \right.$$

$$\left. + 2(1.661 + 1.2851) \right] = 9.7203$$

Thus $\int_2^8 \frac{dx}{\log_{10} x} = \mathbf{9.7203}$

Note : *The function* $1/\log_{10} x$ *is not integrable by analytical methods.*

---

53. *Use Simpson's 3/8 th rule to obtain the approximate value of* $\int_0^{0.3} (1-8x^3)^{1/2}\, dx$ *by considering 3 equal intervals.*

>> The length of each interval $(h) = \dfrac{0.3-0}{3} = 0.1$;∼ $n = 3$.

The values of x and $y = (1-8x^3)^{1/2}$ are tabulated.

x	0	0.1	0.2	0.3
$y = (1-8x^3)^{1/2}$	1	0.996	0.9675	0.8854
	y_0	y_1	y_2	y_3

Simpson's 3/8 th the rule for $n = 3$ is given by

$$\int_a^b y\, dx = \frac{3h}{8}\left[(y_0+y_3)+3(y_1+y_2)\right]$$

$$\therefore \quad \int_0^{0.3} (1-8x^3)^{1/2}\, dx = \frac{3(0.1)}{8}\left[(1+0.8854)+3(0.996+0.9675)\right]$$

Thus $\displaystyle\int_0^{0.3} (1-8x^3)^{1/2} = \mathbf{0.2916}$

---

54. *Use Simpson's 1/3 rd rule to find* $\int_0^{0.6} e^{-x^2}\, dx$ *by taking 6 sub- intervals.*

>> Length of each subinterval $(h) = \dfrac{0.6-0}{6} = 0.1$; $n = 6$

The values of x and $y = e^{-x^2}$ correct to four decimal places are tabulated.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$y = e^{-x^2}$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's 1/3 rd rule for $n = 6$ is given by

$$\int_a^b y\, dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$\therefore \int_0^{0.6} e^{-x^2}\, dx = \frac{0.1}{3}\Big[(1+0.6977) + 4(0.99+0.9139+0.7788) + 2(0.9608+0.8521) \Big] = 0.5351$$

Thus $\displaystyle\int_0^{0.6} e^{-x^2}\, dx = \mathbf{0.5351}$

Note : e^{-x^2} *is not integrable by analytical methods.*

---

56. *Evaluate* $\displaystyle\int_0^1 \frac{x\,dx}{1+x^2}$ *by Weddle's rule taking seven ordinates and hence find* $\log_e 2$

>> Seven ordinates means that the interval [0, 1] must be divided into 6 equal parts.

Length of each part $(h) = \frac{1-0}{6} = \frac{1}{6}$; $n = 6$.

The points of division are $x = 0, \frac{1}{6}, \frac{2}{6} = \frac{1}{3}, \frac{3}{6} = \frac{1}{2}, \frac{4}{6} = \frac{2}{3}, \frac{5}{6}, \frac{6}{6} = 1$

and the corresponding values of $y = x/1+x^2$ are computed.

$$x = 0, \quad y = \frac{0}{1+0^2} = 0 \quad (y_0)$$

$$x = \frac{1}{6}, \quad y = \frac{1/6}{1+(1/6)^2} = \frac{6}{37} \quad (y_1)$$

$$x = \frac{1}{3}, \quad y = \frac{1/3}{1+(1/3)^2} = \frac{3}{10} \quad (y_2)$$

$$x = \frac{1}{2}, \quad y = \frac{1/2}{1+(1/2)^2} = \frac{2}{5} \quad (y_3)$$

$$x = \frac{2}{3}, \quad y = \frac{2/3}{1+(2/3)^2} = \frac{6}{13} \quad (y_4)$$

$$x = \frac{5}{6}, \quad y = \frac{5/6}{1+(5/6)^2} = \frac{30}{61} \quad (y_5)$$

$$x = 1, \quad y = \frac{1}{1+(1)^2} = \frac{1}{2} \quad (y_6)$$

Weddle's rule for $n = 6$ is given by,

$$\int_a^b y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 \right]$$

$$\int_0^1 \frac{x\,dx}{1+x^2} = \frac{3}{10}\cdot\frac{1}{6}\left[0 + 5\left(\frac{6}{37}\right) + \frac{3}{10} + 6\left(\frac{2}{5}\right) + \frac{6}{13} + 5\left(\frac{30}{61}\right) + \frac{1}{2} \right] \approx 0.3466$$

Thus $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx = \mathbf{0.3466}$

Now we shall deduce the value of $\log_e 2$.

Integrating theoretically the given function we have,

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x\,dx}{1+x^2}$$

$$= \frac{1}{2}\log_e(1+x^2)\Big]_0^1 = \frac{1}{2}\log_e 2 - \frac{1}{2}\log_e 1$$

$$\therefore \quad \int_0^1 \frac{x}{1+x^2} = \frac{1}{2}\log_e 2, \quad \text{since } \log_e 1 = 0$$

Equating this with the numerical value obtained we have,

$$\frac{1}{2}\log_e 2 = 0.3466$$

Thus $\mathbf{\log_e 2 = 0.6932}$

---

56. *Evaluate* $\displaystyle\int_0^{\pi/2} \cos x\,dx$ *by applying Simpson's 1/3rd rule taking eleven ordinates. Compare the value with the theoretical value.*

>> Eleven ordinates means that $[0, \pi/2]$ must be divided into ten equal parts.

The length of each part $(h) = \dfrac{\pi/2 - 0}{10} = \dfrac{\pi}{20} = 9^0$; $n = 10$

The values of x and $\cos x$ correct to four decimal places are tabulated.

x^o	0^o	9^o	18^o	27^o	36^o	45^o	54^o	63^o	72^o	81^o	90^o
$y = \cos x$	1	0.9877	0.9511	0.8910	0.8090	0.7071	0.5878	0.454	0.3090	0.1564	0
.	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Simpson's 1/3rd rule for $n = 10$ is given by

$$\int_a^b y\,dx = \frac{h}{3}\Big[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\Big]$$

$$\int_0^{\pi/2} \cos x\,dx = \frac{1}{3}\cdot\frac{\pi}{20}\Big[(1+0)+4(0.9877+0.8910+0.7071+0.454+0.1564) + 2(0.9511+0.8090+0.5878+0.3090)\Big]$$

ie., $\int_0^{\pi/2} \cos x\ dx = 1.000000358$ by taking $\pi = 22/7$

Thus $\int_0^{\pi/2} \cos x\,dx = 1$

Theoretical value :

$$\int_0^{\pi/2} \cos x = [\sin x]_0^{\pi/2} = \sin(\pi/2) - \sin 0 = 1 - 0 = 1$$

Remark : *Observe accuracy in the numerical value, since the number of strips / parts are more.*

---

57. *Evaluate* $\int_4^{5.2} \log_e x$ *taking 6 equal strips by applying Weddle's rule.*

>> The length of each strip $(h) = \frac{5.2-4}{6} = 0.2$; $n = 6$

The values of x and $y = \log_e x$ are tabulated.

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$y = \log_e x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487
.	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Weddle's rule for $n = 6$ is given by

$$\int_a^b y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$\therefore \int_4^{5.2} \log_e x\,dx = \frac{3(0.2)}{10}\left[1.3863 + 5(1.4351) + 1.4816 + 6(1.5261) + 1.5686 + 5(1.6094) + 1.6487\right]$$

Thus $\int_4^{5.2} \log_e x = 1.8279$

---

58. *Use Weddle's rule to compute the area bounded by the curve $y = f(x)$, x-axis and the extreme ordinates from the following table.*

x	0	1	2	3	4	5	6
y	0	2	2.5	2.3	2	1.7	1.5

>> Area $(A) = \int_0^6 y\,dx$.

We have by data,

$$y_0 = 0,\ y_1 = 2,\ y_2 = 2.5,\ y_3 = 2.3,\ y_4 = 2,\ y_5 = 1.7,\ y_6 = 1.5$$

Weddle's rule for $n = 6$ is given by,

$$\int_a^b y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$A = \int_0^6 y\,dx = \frac{3(1)}{10}\left[0 + 5(2) + 2.5 + 6(2.3) + 2 + 5(1.7) + 1.5\right] = 11.49$$

Thus **the required area is 11.49 sq units.**

---

59. *A plane area is bounded by a curve, the x axis and two extreme ordinates. The area is divided into six figures by equidistant ordinates 2 inches apart, the heights of the ordinates being* 21.65, 21.04, 20.35, 19.61, 18.75, 17.80 and 16.75 *respectively. Find the approximate value of the area by numerical integration.*

>> Here $n = 6$, $h = 2$ and the values of y are :

$y_0 = 21.65, y_1 = 21.04, y_2 = 20.35, y_3 = 19.61, y_4 = 18.75, y_5 = 17.8, y_6 = 16.75$

We shall apply Weddle's rule.

$$\int_a^b y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 \right]$$

$$\text{Area} = \frac{3(2)}{10}\left[21.65 + 5(21.04) + 20.35 + 6(19.61) + 18.75 + 5(17.8) + 16.75 \right]$$

Thus the **required area = 233.616 sq. inches.**

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60. *Find the distance travelled by a train between* 8.20 A.M *and* 9 A.M *from the following data.*

Time	8.20 A.M	8.30 A.M	8.40 A.M	8.50 A.M	9 A.M
Speed in miles/hour	24.2	35	41.3	42.8	39.4

>> If s be the distance travelled and v is the velocity, we know that $v = \frac{ds}{dt}$

$$\therefore \qquad s = \int_{t_1}^{t_2} v\,dt.$$

Since we have a set of five values,

$$n = 4 \text{ and time interval } = 10 \text{ minutes } = \frac{1}{6} \text{ hour } \therefore\ h = \frac{1}{6}$$

Let $v_0 = 24.2,\ v_1 = 35,\ v_2 = 41.3,\ v_3 = 42.8,\ v_4 = 39.4$

Simpson's 1/3 rd rule for $n = 4$ is given by

$$s = \frac{h}{3}\left[(v_0 + v_4) + 4(v_1 + v_3) + 2v_2 \right]$$

$$= \frac{1}{3}\cdot\frac{1}{6}\left[(24.2 + 39.4) + 4(35 + 42.8) + 2(41.3) \right] = 25.4$$

Thus **the distance travelled is 25.4 miles approximately.**

EXERCISES

Evaluate by using simpson's 1/3 rd rule :

1. $\int_0^{\pi/2} \sqrt{\sin x}\,dx$ by taking $n = 10$

2. $\int_0^1 e^{-x^2}\,dx$ by taking $h = 0.1$.

3. $\int_0^1 \frac{\log(1+x^2)}{1+x^2}\,dx$ by taking $n = 10$

4. $\int_0^1 \frac{x^2}{1+x^3}\,dx$ by taking 7 ordinates and hence deduce the value of $\log_e 2$.

Evaluate by using Simpson's 3/8 th rule :

5. $\int_0^{\pi/2} e^{\sin x}\,dx$

6. $\int_0^{0.3} (1-8x^3)^{3/2}\,dx$ taking seven ordinates.

7. $\int_1^2 \frac{dx}{\sqrt{3+2x-x^2}}$ by taking six equal strips and hence deduce the value of π.

Evaluate the following definite integrals by dividing the given interval into six equal parts applying (a) Simpson's 1/3rd rule (b) Simpson's 3/8th rule :

8. $\int_0^1 \frac{dx}{(1+x)^2}$ 9. $\int_0^{\pi} \frac{dx}{2+\cos x}$ 10. $\int_0^1 \frac{x\,dx}{1+x^2}$

11. A rocket is launched from the ground. Its acceleration a is registered during the first one minute and is given below. Apply the Simpson's 3/8th rule to find the velocity of the rocket at the first minute.

Time *(in seconds)*	0	10	20	30	40	50	60
Acceleration (cm/sec^2)	30	31.63	33.34	35.47	37.75	40.33	43.25

Hint : velocity $v = \int a\,dt$, since $a = \frac{dv}{dt}$

Evaluate by using Weddle's rule :

12. $\int_0^6 \frac{dx}{1+x^2}$ by taking $h = 1/2$

13. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ by taking six equal strips.

Also (a) find the theoretical value (b) deduce the value of π

14. $\int_{-\pi/2}^{\pi/2} \cos x \; dx$ correct to two decimal places by taking.

(a) Six sub-intervals (b) Twelve sub-intervals

Evaluate the following definite integrals by dividing the given interval into six equal parts applying (a) Simpson's $1/3^{rd}$ rule (b) Simpson's $3/8^{th}$ rule (c) Weddle's rule. Also compare the answers with the theoretical value.

15. $\int_0^1 \frac{dx}{(1+x)^2}$

16. $\int_0^{\pi} \frac{dx}{2+\cos x}$

ANSWERS

1. 1.1873

2. 0.7468

3. 0.173

4. 0.23105 ; 0.69315

5. 3.1016 (for $n = 3$)

6. 0.2765

7. 0.5236 ; 3.1416

8. 0.500908 ; 0.5001895

9. 1.8133 ; 1.8114

10. 0.3465 ; 0.3466

11. 2150.025 cms/sec^2

12. 1.377

13. 0.7854, $\pi/4$, 3.1416

14. 1.29, 1.81

15. 0.500908, 0.5001895, 0.5000119 ; theoretical value = 0.5

16. 1.8133, 1.8114, 1.8148 ; theoretical value = $\pi/\sqrt{3}$ = 1.8138